SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #10 due 06/28

Problem 1. The Sobolev space $H^s(\mathbb{R})$ for $s \in \mathbb{R}$ is defined as

$$H^s(\mathbb{R}^d) = \{u \in \mathscr{S}'(\mathbb{R}^d) \ : \ (1+|\xi|^2)^{s/2}\hat{u} \in L_2(\mathbb{R}^d)\}$$

with norm

$$\|u\|^2_{H^s(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\, d\xi \ .$$

a.) Prove that $\hat{u} \in L_1(\mathbb{R}^d)$ implies $u \in C(\mathbb{R}^d)$.
b.) Prove that $u \in H^s(\mathbb{R}^d)$ for some $s > d/2$ implies that $u \in C(\mathbb{R}^d)$ and the estimate $\sup_{\mathbb{R}^d}|u| \leq C\|u\|_{H^s(\mathbb{R}^d)}$. This statement is know as *Sobolev imbedding theorem.*

Problem 2. Suppose that Ω is bounded and open and that $\mu_1 \leq \mu_2, \ldots$ are the eigenvalues of the Dirichlet Laplacian.
a.) Prove that

$$\mu_1 = \min\left\{\int_\Omega |\nabla u|^2\, dx \ : \ u \in \mathring{H}^1(\Omega), \|u\|_{L_2(\Omega)} = 1\right\}$$

b.) Denote $S_k = \mathrm{span}[u_1, \ldots, u_k]$ where the $u_j \in \mathring{H}^1(\Omega)$ is the eigenfunction of the Dirichlet Laplacian corresponding to the eigenvalue μ_j for $j = 1, \ldots, k$, and by $S_k^\perp$ its orthogonal complement in $\mathring{H}^1(\Omega)$. Prove that

$$\mu_{k+1} = \min\left\{\int_\Omega |\nabla u|^2\, dx \ : \ u \in S_k^\perp, \|u\|_{L_2(\Omega)} = 1\right\}$$

for $k = 1, 2, \ldots$.

Problem 3. Prove the monotonicity of the Dirichlet Laplacian with respect to the domain. More precicely, if D is a second open and bounded set and $D \subset \Omega$ show that $\mu_k(\Omega) \leq \mu_k(D)$ for $k = 1, 2, \ldots$.