## SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
## LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

**Homework #10** Solutions

**Problem 1.** The Sobolev space $H^s(\mathbb{R})$ for $s \in \mathbb{R}$ is defined as

$$H^s(\mathbb{R}^d) = \{u \in \mathscr{S}'(\mathbb{R}^d) \ : \ (1+|\xi|^2)^{s/2}\hat{u} \in L_2(\mathbb{R}^d)\}$$

with norm

$$\|u\|^2_{H^s(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\,d\xi\,.$$

a.) Prove that $\hat{u} \in L_1(\mathbb{R}^d)$ implies $u \in C(\mathbb{R}^d)$.

*Proof.* Suppose that $\{x_k\} \subset \mathbb{R}^d$ is a sequence converging to $x \in \mathbb{R}^d$. Then

$$|u(x) - u(x_k)| \le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|e^{ix\cdot\xi} - e^{ix_k\cdot\xi}||\hat{u}(\xi)|d\xi \longrightarrow 0$$

where we used the triangle inequality and the Lebesgue dominated convergence theorem. □

b.) Prove that $u \in H^s(\mathbb{R}^d)$ for some $s > d/2$ implies that $u \in C(\mathbb{R}^d)$ and the estimate $\sup_{\mathbb{R}^d}|u| \le C\|u\|_{H^s(\mathbb{R}^d)}$. This statement is know as *Sobolev imbedding theorem.*

*Proof.* Note that

$$\sup_{x\in\mathbb{R}}|u(x)| = \sup_{x\in\mathbb{R}}\frac{1}{(2\pi)}\left|\int_{\mathbb{R}} e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi\right| \le \|\hat{u}\|_{L_1(\mathbb{R}^d)}\,.$$

Furthermore, by Hölder's inequality

$$\int_{\mathbb{R}^d}|\hat{u}(\xi)|\,d\xi = \left(\int_{\mathbb{R}^d}|\hat{u}(\xi)|^2(1+|\xi|^2)^s\,d\xi\right)^{1/2}\left(\int_{\mathbb{R}^d}(1+|\xi|^2)^{-s}\,d\xi\right)^{1/2}$$

The last integral is convergent if $s > d/2$. This can be shown by introducing polar (spherical) coordinates in $\mathbb{R}^d$. □

**Problem 2.** Suppose that $\Omega$ is bounded and open and that $\mu_1 \le \mu_2, \ldots$ are the eigenvalues of the Dirichlet Laplacian.
a.) Prove that

$$\mu_1 = \min\left\{\int_\Omega|\nabla u|^2\,dx \ : \ u \in \overset{\circ}{H}{}^1(\Omega), \|u\|_{L_2(\Omega)} = 1\right\}$$

*Proof.* Let $u \in \mathring{H}^1(\Omega)$ such that $\|u\|_{L_2(\Omega)} = 1$. According to Theorem 6.12, there exists an orthonormal basis ( in $L_2(\Omega)$ of eigenfunctions $u_k \in \mathring{H}^1(\Omega)$ for $k = 1, 2, ...$ to the Dirichlet Laplacian $-\Delta : \mathring{H}^1(\Omega) \to H^{-1}(\Omega)$. There exist numbers $\alpha_1, \alpha_2, ... \in \mathbb{R}$ such that

$$u = \sum_{k=1}^{\infty} \alpha_k u_k$$

and

$$\sum_{k=1}^{\infty} \alpha_k^2 = 1 \ . \tag{1}$$

The eigenfunctions satisfy the identity

$$\int_\Omega |\nabla u_k|^2 \, dx = \mu_k \int_\Omega |u_k|^2 \, dx$$

Hence, the functions $u_k/\sqrt{\mu_k}$ are an orthonormal set in $\mathring{H}^1(\Omega)$ and one can show that they form an orthonormal basis. Then there exist number $\beta_1, \beta_2, ... \in \mathbb{R}$ such that

$$u = \sum_{k=1}^{\infty} \beta_k \frac{u_k}{\sqrt{\mu_k}} \ .$$

Hence, $\beta_l = \sqrt{\mu_l}\alpha_l$ for $l = 1, 2, ...$ and the series $\sum_{l=1}^{\infty} \mu_l \alpha_l^2$ is convergent. Using Parseval's identity gives

$$\int_\Omega |\nabla u|^2 \, dx == \sum_{j=1}^{i} nfty \beta_j^2 = \sum_{j=1}^{\infty} \mu_j \alpha_j^2 \ ,$$

and since the eigenvalues are increasing we know that the sum on the right-hand side is greater or equal to $\mu_1$ because of (1). On the other hand, if $u = u_1$, then $\int_\Omega |\nabla u_1| \, dx = 1$. □

b.) Denote $S_k = \text{span}[u_1, ..., u_k]$ where the $u_j \in \mathring{H}^1(\Omega)$ is the eigenfunction of the Dirichlet Laplacian corresponding to the eigenvalue $\mu_j$ for $j = 1, .., k$ and by $S_k^\perp$ its orthogonal complement in $\mathring{H}^1(\Omega)$. Prove that

$$\mu_{k+1} = \min \left\{ \int_\Omega |\nabla u|^2 \, dx \ : \ u \in S_k^\perp, \|u\|_{L_2(\Omega)} = 1 \right\}$$

for $k = 1, 2, ....$

*Proof.* Now $u = \sum_{j=k+1}^{\infty} \alpha_j u_j$ with $\sum_{j=k+1}^{\infty} \alpha_j^2 = 1$. Then, using the same approach as in the first part of the problem

$$\int_\Omega |\nabla u|^2 \, dx = \sum_{j=k+1}^{\infty} \mu_j \alpha_j^2 \geq \mu_{k+1} \ ,$$

and equality holds for $u = u_{k+1}$. □

**Problem 3.** Prove the monotonicity of the Dirichlet Laplacian with respect to the domain. More precicely, if $D$ is a second open and bounded set and $D \subset \Omega$ show that $\mu_k(\Omega) \leq \mu_k(D)$ for $k = 1, 2, ....$

*Proof.* Using problem 2 we know that

$$\mu_1(D) = \min\left\{\int_D |\nabla u|^2\, dx \;:\; u \in \mathring{H}^1(D), \|u\|_{L_2(D)} = 1\right\}$$

and

$$\mu_1(\Omega) = \min\left\{\int_\Omega |\nabla u|^2\, dx \;:\; u \in \mathring{H}^1(\Omega), \|u\|_{L_2(\Omega)} = 1\right\}\ .$$

Note that every function $u \in \mathring{H}^1(D)$ can be extended to $\Omega \setminus D$ by zero and becomes then a function in $\mathring{H}^1(\Omega)$. Using this extension we have the inclusion

$$\left\{u \in \mathring{H}^1(D), \|u\|_{L_2(D)} = 1\right\} \subset \left\{u \in \mathring{H}^1(\Omega), \|u\|_{L_2(\Omega)} = 1\right\}$$

and $\mu_1(\Omega) \leq \mu_1(D)$ follows.

From problem 2 we recall that

$$\mu_{k+1}(D) = \min\left\{\int_D |\nabla u|^2\, dx \;:\; u \in S_k^\perp(D), \|u\|_{L_2(D)} = 1\right\}$$

for $k = 1, 2, \ldots$ where $S_k(D) = \mathrm{span}[u_1, \ldots., u_k]$, the linear span of the first $k$ eigenfunctions of the Dirichlet Laplacian in $D$. Extending those function by zero to all of $\Omega$, they can be considered as eigenfunctions of the Dirichlet-Laplacian in $\Omega$. However, there is no guarantee that for example the function $u_2$ is the eigenfunction corresponding to the eigenvalue $\mu_2(\Omega)$, it may belong to a bigger eigenvalue. This argument gives the desired conclusion. $\square$