SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #11 Solutions

Problem 1. Suppose that $K : H \to H$ is a compact linear operator on a Hilbert space H. Prove that
a.) $N(I-K) = \{x \in H \ : \ Kx = x\}$ is finite-dimensional.

Proof. We argue by contradiction. Suppose that $\{x_1, x_2, ...\}$ is a sequence of orthonormal unit vectors such that $Kx_j = x_j$ for $j = 1, 2, ...$ Then $\|Kx_j - Kx_l\| = \|x_j - x_l\| = \sqrt{2}$ for all $l \neq k$. On the other hand, the sequence $\{x_j\}_{j=1}^{\infty}$ is bounded and hence the sequence Kx_j has a convergent subsequence. However, this is not possible since $\|Kx_j - Kx_l\| = \sqrt{2}$. □

b.) $R(I-K) = \{x - Kx \ : \ x \in H\}$ is closed. Hint: Show at first that there exists a constant $\gamma > 0$ such that

$$\|u - Ku\| \geq \gamma \|u\| \quad \text{for all } u \in N(I-K)^{\perp} .$$

Proof. Again, we argue by contradiction. Suppose the estimate does not hold. Then there exists a sequence $\{x_j\}_{j=1}^{\infty}$ of unit vectors such that

$$\|x_j - Kx_j\| < \frac{1}{j} \quad \text{for } j = 1, 2, ...$$

which implies

$$Kx_j - x_j \to 0 \tag{1}$$

in H. Since the sequence $\{x_j\}$ is bounded, there exists a weakly convergent subsequence which we denote for simplicity again by $\{x_j\}$, that is $x_j \rightharpoonup x \in H$. Since K is compact, we have $Kx_j \to Kx$ in H and because of (1) one obtains $x_j \to x$ in H. Consequently, we have $x = Kx$ which gives $x \in N(I-K)$ and

$$(x_j, x) = 0 \quad \text{for } k = 1, 2, ...$$

Letting $j \to \infty$ gives $\|x\| = 0$ which is a contradiction to $\|x_j\| = 1$ for all $j \in \mathbb{N}$.

Suppose now that $y_j \in R(I-K)$ for $j = 1, 2, ...$ and that $y_j \to y \in H$. We have $y_j = x_j - Kx_j$ for some $x_j \in H$, $j = 1, 2, ...$ and we need to find a $x \in H$ such that $y = x - Kx$. Without loss of generality we may assume that $x_j \in N(I-K)^{\perp}$. Otherwise we can take the orthogonal projection of x_j onto $N(I-K)$ and subtract it from x_j. Then we know from the first part of the proof that

$$\|x_j - x_l\| \leq \frac{1}{\gamma}\|x_j - x_l - K(x_j - x_l)\| = \frac{1}{\gamma}\|y_j - y_l\| \quad \text{for all } l, j \in \mathbb{N} .$$

This shows that $\{x_j\}$ is a Cauchy sequence and by completeness of H we have $x_j \to x \in H$ and then also $y = x - Kx$. □

Problem 2. For $f \in L_2(\mathbb{R}^d)$ use the Fourier transform in space to derive a solution formula for the initial value problem to the heat equation

$$\begin{aligned} u_t - \Delta u = 0 &\quad \text{in } (0,\infty) \times \mathbb{R}^d \,, \\ u(0,\cdot) = f &\quad \text{in } \mathbb{R}^d \,. \end{aligned}$$

Solution. Set $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx$ and $\hat{u}(t,\xi) = \int_{\mathbb{R}^d} e^{-x\cdot\xi} u(t,x)\,dx$ where the first Fourier transform is to be understood in the L_2 sense and the second one is just a formal one in order to transform the PDE into an ODE with a parameter. On obtains

$$\begin{aligned} \hat{u}_t + |\xi|^2 \hat{u} = 0 &\quad \text{in } (0,\infty) \times \mathbb{R}^d \,, \\ \hat{u}(0,\cdot) = \hat{f} &\quad \text{in } \mathbb{R}^d \,. \end{aligned}$$

The solution to this IVP is given by

$$\hat{u}(t,\xi) = e^{-|\xi|^2 t} \hat{f}(\xi)$$

Note that $\hat{u}(t,\cdot) \in L_2(\mathbb{R}^d)$ for all $t \in (0,\infty)$. Actually, one even has $\hat{(}t,\xi) \in \mathscr{S}(\mathbb{R}^d)$ for all $t > 0$. Hence, one can invert the Fourier transform and obtains

$$u(t,x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{f}(\xi)\,d\xi \,.$$

Now, since

$$\mathscr{F}^{-1}(e^{-|\xi|^2 t}) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)} \quad \text{or} \quad \mathscr{F}\left(\frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)}\right) = e^{-|\xi|^2 t}$$

and conclude that

$$u(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)} * f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/(4t)} f(y)\,dy \,.$$

Note that this formula coincides with the one given in Theorem 7.3. It is also possible to give a regularity statement of the solution with L_2 initial data. In particular one concludes with the aid of Parseval's identity that $u(t,\cdot) \in L_2(\mathbb{R}^d)$ for all $t > 0$ and that $\|u(t,\cdot)\|_{L_2(\mathbb{R}^d)} \le \|f\|_{L_2(\mathbb{R}^d)}$.

Problem 3. Suppose that $u \in H^s(\mathbb{R}^d)$ for some $s > 1/2$. Show that the mapping $T : C_0^\infty(\mathbb{R}^d) \to C_0^\infty(\mathbb{R}^{d-1})$ given by $Tu(x', x_d) = u(x', 0)$ extends to a continuous linear operator from $H^s(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^{d-1})$.

Proof. Suppose that $u \in C_0^\infty(\mathbb{R}^d)$ and let $g = Tu \in C_0^\infty(\mathbb{R}^{d-1})$. Note at first that

$$g(x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix'\cdot\xi'} \hat{u}(\xi) d\xi = \frac{1}{(2\pi)^{(d-1)}} \int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi) d\xi_d \right\} d\xi'$$

which, after applying the Fourier transform with respect to the x' variable, results in

$$\hat{g}(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi)\,d\xi_d \,.$$

For brevity we set $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. Then, using the Cauchy-Schwarz inequality

$$\begin{aligned}
\|g\|^2_{H^{1/2}(\mathbb{R}^{d-1})} &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d+1}} \langle \xi' \rangle |\hat{g}(\xi')|^2 \, d\xi' \\
&= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \langle \xi' \rangle^{s-1/2} \left| \int_{\mathbb{R}} \hat{u}(\xi) d\xi_d \right|^2 d\xi' \\
&\leq \frac{1}{(2\pi)^{d+1}} \int \langle \xi' \rangle^{s-1/2} \left\{ \int_{\mathbb{R}} |\hat{u}(\xi)|^2 \langle \xi \rangle^s d\xi_d \int_{\mathbb{R}} \langle \xi \rangle)^{-s} d\xi_d \right\} d\xi' \\
&= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d-1}} \left\{ \int_{\mathbb{R}} |\hat{u}(\xi)|^2 \langle \xi \rangle^s d\xi_d \int_{\mathbb{R}} \frac{\langle \xi' \rangle^{s-1/2}}{\langle \xi \rangle^s} d\xi_d \right\} d\xi' \, .
\end{aligned}$$

Now one computes the integral using the substitution $z = \xi_d / \sqrt{1 + |\xi'|^2}$ and observes that

$$\int_{\mathbb{R}} \frac{(1 + |\xi'|^2)^{s-1/2}}{(1 + |\xi|^2)^s} d\xi_d = \int_{\mathbb{R}} \frac{(1 + |\xi'|^2)^{s-1/2}}{(1 + |\xi'|^2 + \xi_d^2)^s} d\xi_d = \int_{\mathbb{R}} \frac{dz}{(1 + |z|^2)^s} = C_s < \infty$$

for $s > 1/2$. One concludes $\|g\|^2_{H^{1/2}(\mathbb{R}^{d-1})} \leq C \|u\|^2_{H^1(\mathbb{R}^d)}$ for some positive constant C depending on s. Using a density argument this shows that the operator T extends to a continuous, linear operator from $H^1(\mathbb{R}^d)$ into $H^{1/2}(\mathbb{R}^{d-1})$. $\square$