SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #12 Solutions

Problem 1. Suppose that $K : H \to H$ is a compact linear operator on a Hilbert space H and K^* is the adjoint. Prove the following statements.
a.) $N(I-K^*)^\perp = R(I-K)$. Hint: Make use of Problem 1 from the previous week.

Proof. Recall that $R(I-K)$ is closed, see HW #11, problem 1. Hence, it will suffice to show $N(I-K^*) = R(I-K)^\perp$.

Choose $z \in H$. Then $x = z - Kz \in R(I-K)$. Suppose now that $y \in R(I-K)^\perp$, that is $(y,x) = 0$. This implies

$$0 = (y, z - Kz) = (y,z) - (y,Kz) = (y,z) - (K^*y, z) = (y - K^*y, z) \qquad \text{for all } z \in H \ .$$

Hence we have shown that $y \in N(I-K^*)$ and thus $R(I-K)^\perp \subset N(I-K^*)$.

To show the opposite inclusion, suppose that $x \in N(I-K^*)^\perp$, that is

$$(x, (I-K)z) = ((I-K^*)x, z) = 0 \qquad \text{for all } z \in H \ .$$

Hence, $x \in R(I-K)^\perp$ and thus $N(I_K^*) \subset R(I-K)^\perp$. ☐

b.) $R(I-K) = H$ if and only if $N(I-K) = \{0\}$.

Proof. "$\Longleftarrow$" We argue by contradiction. Suppose $N(I-K) = \{0\}$ and that $H_1 := R(I-K) \subsetneq H$. According to last week's homework problem, H_1 is a closed subspace. Define now inductively a sequence of closed subspaces $H_{k+1} = (I-K)H_k$ for $k = 1, 2, ...$ Note that $H_{k+1} = (I-K)^k(I-K)H \subset H_k$ and $H_{k+1} \subsetneq H_k$ since $N(I-K) = \{0\}$.

Choose now unit vectors $x_k \in H_k$ such that $x_k \in H_{k+1}^\perp$. Then, for all $l, k \in \mathbb{N}$,

$$Kx_k - Kx_l = -(x_k - Kx_k) + (x_l - Kx_l) + x_k - x_l$$

and we observe $x_k - Kx_k \in H_{k+1}$, $x_l - Kx_l \in H_{l+1}$, $x_k \in H_k$, and $x_l \in H_{l+1}^\perp$. For $k > l$ we have because of the inclusion $H_{k+1} \subset H_k \subset H_{l+1}$ that

$$\|Kx_k - Kx_l\| = \| -(x_k - Kx_k) + (x_l - Kx_l) + x_k\| + \|x_l\| \geq 1$$

This is a contradiction to the compactness of K since it prevents the existence of a converging subsequence.

"$\Longrightarrow$" Suppose that $R(I-K) = H$. Then, by part a.) we know that $N(I-K^*) = \{0\}$ and by the first part of this proof gives $R(I-K^*) = H$. Use part a.) with K replaced by K^* to conclude that $N(I-K) = \{0\}$. ☐

Problem 2. For $f \in L_2(\mathbb{R}^d)$ use the Fourier transform in space to derive a solution formula for the initial value problem to the Schrödinger equation

$$iu_t - \Delta_x u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d \ ,$$
$$u(0,\cdot) = f \quad \text{in } \mathbb{R}^d \ .$$

Here $i = \sqrt{-1}$.

Solution. After Fourier transform in x one obtains the initial value problem

$$i\hat{u}_t(t,xi) = -|\xi|^2\hat{u}(t,\xi) \quad t \in [0,\infty), \quad \hat{u}(0,\xi) = \hat{f}(\xi) \ , \qquad \text{for all } \xi \in \mathbb{R}^d \ .$$

The solution to this problem is given by

$$\hat{u}_t = e^{-it|\xi|^2}\hat{f}(\xi) \ ,$$

and using the inverse Fourier transform, one obtains the formula

$$u(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{-it|\xi|^2}\hat{f}(\xi)\,d\xi$$

where we note that $u(t,\cdot) \in L_2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$, since $f \in L_2(\mathbb{R}^d)$. The inverse Fourier transform can be computed using the formula

$$u(t,x) = \mathscr{F}^{-1}[e^{-it|\xi|^2}](t,x) * f(x) \ .$$

The amazing fact is that formula discussed last week (07/05/2016)

$$\mathscr{F}[e^{-a|x|^2}] = \left(\frac{\pi}{a}\right)^{d/2} e^{-|\xi|^2/(4a)} \qquad a > 0$$

is even true all $a \in \mathbb{C}$ satisfying $\Re a \geq 0$ and $a \neq 0$. Hence, one obtains choosing $a = 1/(4ti)$ that

$$\mathscr{F}^{-1}[e^{-it|\xi|^2}](t,x) = \frac{1}{(4\pi it)^{d/2}}e^{-|x|^2/(4ti)} \ ,$$

whence

$$u(t,x) = \frac{1}{(4\pi it)^{d/2}}\int_{\mathbb{R}^d} e^{-|x-y|^2/(4ti)}f(y)\,dy \ .$$

Problem 3. Find a fundamental solution to the Schrödinger operator, that is a distribution $\Phi(t,x) \in \mathscr{D}'(\mathbb{R}^{d+1})$ that satisfies

$$i\partial_t\Phi = \Delta_x\Phi \text{ for } t \neq 0, x \in \mathbb{R}^d \quad \text{and} \quad \Phi(0,x) = \delta_0 \text{ for } x \in \mathbb{R}^d \ .$$

Hint: Use problem 2 and Lemma 7.2, which also explains how to understand the condition $\Phi(0,x) = \delta_0$.

Solution. Set

$$\Phi(t,x) = \begin{cases} \dfrac{1}{(4\pi it)^{d/2}}e^{-|x|^2/(4ti)} & \text{for} \quad t > 0 \ , \\ 0 & \text{for} \quad t \leq 0 \ . \end{cases}$$

We will show that $\Phi(t_k,\cdot) \to \delta_0$ as $t_k \to 0^+$. Choose a sequence $t_k \geq 0$ for $k = 1,2,...$ such that $t_k \to 0$. From problem 2 we know that

$$\mathscr{F}[\Phi(t_k,\cdot)](\xi) = e^{it_k|\xi|^2}$$

and given $u \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\Phi(t_k,\cdot)(\hat{u}) = \int_{\mathbb{R}^d} e^{it_k|\xi|^2}u(\xi)\,d\xi \to \int_{\mathbb{R}^d} u(\xi)\,d\xi = \hat{u}(0) \quad \text{as } k \to \infty \ .$$