SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #13 due 07/19

Problem 1. Using the Fourier transform in x, solve the initial value problem for the wave equation in the whole space $\mathbb{R}^d$

$$\begin{aligned} u_{tt} - c^2 \Delta u = 0 \quad &\text{in } (0,T) \times \mathbb{R}^d \ , \\ u(0,x) = f(x) \quad &x \in \mathbb{R}^d \ , \\ u_t(0,x) = g(x) \quad &x \in \mathbb{R}^d \ . \end{aligned}$$

Assuming that f satisfies $\nabla f \in L_2(\mathbb{R}^d)$ and that $g \in L_2(\mathbb{R}^d)$, what can you say about the regularity of the solution ?

Problem 2. Consider the following initial-boundary value problem for the heat equation on the interval $(0, \pi)$, that is

$$\begin{aligned} \partial_t u = \partial^2_{xx} u \quad &(t,x) \text{ in } [0,\infty) \times (0,\pi) \ , \\ u(0,x) = f(x) \quad &x \in (0,\pi) \ , \\ u(t,0) = u(t,\pi) = 0 \quad &t \in (0,\infty) \ . \end{aligned}$$

Given $f \in L_2(0,\pi)$ follow the proof of Theorem 8.10 to construct an infinite series solution to this problem. In this case you can work with the eigenfunctions of the operator $d^2/dx^2 : \mathring{H}^1(0,\pi) \to H^{-1}(0,\pi)$ and use the fact that the odd extension of $f \in L_2(0,\pi)$ to the interval $(-\pi,\pi)$ can be expanded into a Fourier sine series.

Problem 3.* Consider the following initial value problem

$$\begin{aligned} w_{tt} - w_{x_1 x_1} - \lambda^2 w = 0 \quad &\text{for } t \in (0,\infty), x_1 \in \mathbb{R} \\ w(0,x_1) = 0 \quad &\text{for all } x_1 \in \mathbb{R} \\ w_t(0,x_1) = \psi(x_1) \quad &\text{for all } x_1 \in \mathbb{R} \end{aligned}$$

Given $\psi \in C^2(\mathbb{R})$ show that the classical solution to this problem is given by

$$w(t,x_1) = \frac{1}{2} \int_{x_1 - t}^{x_1 + t} J_0(\lambda \sqrt{t^2 - (x_1 - y_1)^2}) \psi(y_1) \, dy \ ,$$

where

$$J_0(\lambda) = \frac{2}{\pi} \int_0^{\pi/2} \cos(\lambda \sin z) \, dz \ ,$$

is the Bessel function of order zero. Hint: If w is a solution to the equation above that the function $u(t,x) = \cos(\lambda x_2) w(t,x_1)$ is a solution to the wave equation in $\mathbb{R}_t \times \mathbb{R}^2$. Use then Theorem 9.3 to write a formula for $u(t,x)$ in terms of the initial data and use a substitution in the integral.