SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #13 Solutions

Problem 1. Using the Fourier transform in x, solve the initial value problem for the wave equation in the whole space $\mathbb{R}^d$

$$\begin{aligned} u_{tt} - c^2\Delta u &= 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d \ , \\ u(0,x) &= f(x) \quad x\in\mathbb{R}^d \ , \\ u_t(0,x) &= g(x) \quad x\in\mathbb{R}^d \ . \end{aligned}$$

Assuming that f satisfies $\nabla f \in L_2(\mathbb{R}^d)$ and that $g \in L_2(\mathbb{R}^d)$, what can you say about the regularity of the solution ?

Solution. Using the Fourier transform in x, the initial value problem is transferred to the following initial value problem for a second order ordinary differential equation with parameter ξ,

$$\hat{u}_{tt} + c^2|\xi|^2\hat{u} = 0 \ , \qquad \hat{u}(0,\cdot) = \hat{f} \ , \quad \hat{u}_t(0,\cdot) = \hat{g} \ .$$

The solution to this problem is given by

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos(c|\xi|t) + \hat{g}(\xi)\frac{\sin(c|\xi|t)}{c|\xi|} \ .$$

and inverting the Fourier transfrom gives a solution formula for u. This formula will be the same as the one presented in Theorem 9.1, at least in the case that $d = 3$. However, a statement regarding the regularity can be made at the level of the Fourier transform. If $\nabla f, g \in L_2(\mathbb{R}^d)$, then $\hat{g} \in L_2(\mathbb{R}^d)$ and $|\xi|\hat{f} \in L_2(\mathbb{R}^d)$. Consequently, since $\cos(c|\xi|t)$ and $\sin(c|\xi|t)$ are both bounded by one, one knows that

$$|\xi|\,|\hat{u}(t,\xi)| \leq |\xi|\,|\hat{f}(\xi)| + |\hat{g}(\xi)|$$

which implies that $\nabla_x u(t,\cdot) \in L_\infty(0,\infty;L_2(\mathbb{R}^d))$.

Problem 2. Consider the following initial-boundary value problem for the heat equation on the interval $(0,\pi)$, that is

$$\begin{aligned} \partial_t u &= \partial^2_{xx} u \quad (t,x) \text{ in } [0,\infty)\times(0,\pi) \ , \\ u(0,x) &= f(x) \quad x\in(0,\pi) \ , \\ u(t,0) = u(t,\pi) &= 0 \quad t\in(0,\infty) \ . \end{aligned}$$

Given $f \in L_2(0,\pi)$ follow the proof of Theorem 8.10 to construct an infinite series solution to this problem. In this case you can work with the eigenfunctions of the operator $d^2/dx^2 : \mathring{H}^1(0,\pi) \to H^{-1}(0,\pi)$ and use the fact that the odd extension of $f \in L_2(0,\pi)$ to the interval $(-\pi,\pi)$ can be expanded into a Fourier sine series.

Solution. Note that the eigenfunctions on the operator $-d^2/dx^2$ with zero boundary conditions on the interval $(0, \pi)$ are given by

$$v_n = \sin(nx) \qquad n = 1, 2, ...$$

with corresponding eigenvalue $\lambda_n = n^2$. Following the proof of Theorem 8.10, we set $v_n = \text{span}[v_1, v_2, ..., v_n]$. Furthermore, since $f \in L_2(0, \pi)$ we have the Fourier expansion

$$f = \sum_{n=1}^{\infty} f_n v_n \ , \qquad f_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(n\pi x)\, dx \ ,$$

with the series converging in $L_2(0, \pi)$. Then one builds approximate solutions

$$u_n(t, x) = \sum_{k=1}^{n} \alpha_k(t) v_k(x)$$

which are solutions to the initial value problem for a first-order system with n equations:

$$\left(\sum_{k=1}^{n} \alpha_k'(t) v_k, v_j \right) + \left(\sum_{k=1}^{n} \alpha_k(t) v_k', v_j' \right) = 0 \quad \text{for } j = 1, 2, ..., n \ ,$$

$$\sum_{k=1}^{n} \alpha_k(0) v_k(x) = \sum_{k=1}^{n} f_k v_k \ .$$

Since the $\{v_n\}$ form an orthogonal set, the system decouples, and one obtains n scalar initial value problems

$$\alpha_k'(t) + k^2 \alpha_k(t) = 0 \ , \quad \alpha_k(0) = f_k \ , \qquad k = 1, 2, ..., n \ ,$$

which has the unique solution $\alpha_k = e^{-k^2 t} f_k$. Hence, the approximate solution is

$$u_n(t, x) = \sum_{k=1}^{n} f_k e^{-k^2 t} v_k(x) \ .$$

The proof of Theorem 8.10 shows that the infinite series

$$u(t, x) = \sum_{k=1}^{\infty} f_k e^{-k^2 t} v_k(x)$$

is in $L_2(0, T; \mathring{H}^1(0, \pi)) \cap H^1(0, T; H^{-1}(\Omega))$ and is a weak solution to the initial value problem above.

Problem 3.* Consider the following initial value problem

$$w_{tt} - w_{x_1 x_1} + \lambda^2 w = 0 \quad \text{for } t \in (0, \infty), x_1 \in \mathbb{R} \ ,$$
$$w(0, x_1) = 0 \quad \text{for all } x_1 \in \mathbb{R} \ ,$$
$$w_t(0, x_1) = \psi(x_1) \quad \text{for all } x_1 \in \mathbb{R} \ ,$$

where $\lambda > 0$ is a real parameter. Given $\psi \in C^2(\mathbb{R})$ show that the classical solution to this problem is given by

$$w(t, x_1) = \frac{1}{2} \int_{x_1 - t}^{x_1 + t} J_0(\lambda \sqrt{t^2 - (x_1 - y_1)^2}) \psi(y_1)\, dy_1 \ ,$$

where

$$J_0(\lambda) = \frac{2}{\pi} \int_0^{\pi/2} \cos(\lambda \sin z)\, dz$$

is the Bessel function of order zero. Hint: If w is a solution to the equation above that the function $u(t,x) = \cos(\lambda x_2) w(t, x_1)$ is a solution to the wave equation in $\mathbb{R}_t \times \mathbb{R}^2$. Use then Theorem 9.3 to write a formula for $u(t,x)$ in terms of the initial data and use a substitution in the integral.

Solution. One verifies that the function $u(t,x)$ solves the wave equation in $d = 2$ and that $u(0,x) = 0$ and that $u_t(0,x) = \cos(\lambda x_2)\psi(x_1)$. Hence, using Theorem 9.3

$$u(t,x) = \frac{1}{2\pi} \int_{|x-y|<t} \frac{\cos(\lambda y_2)\psi(y_1)}{\sqrt{t^2 - (x-y)^2}}\, dy$$

Now this integral is evaluated. We have

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi} \int_{x_1-t}^{x_1+t} \int_{x_2-\sqrt{t^2-(x_1-y_1)^2}}^{x_2+\sqrt{t^2-(x_1-y_1)^2}} \frac{\cos(\lambda y_2)\psi(y_1)}{\sqrt{c^2t^2 - (x-y)^2}}\, dy_2 dy_1 \\ &= \frac{1}{2\pi} \int_{x_1-t}^{x_1+t} \int_{-\pi/2}^{\pi/2} \cos(\lambda(x_2 + \sqrt{t^2 - (x_1-y_1)^2} \sin z)) dz dy_1 \end{aligned}$$

where we used the substitution

$$y_2 = (x_2 + \sqrt{t^2 - (x_1-y_1)^2}) \sin z \ , \qquad \frac{dy_2}{\sqrt{t^2 - (x-y)^2}} = dz \ .$$

Using the addition theorem for the cosine function one obtains from here

$$\begin{aligned} u(t,x) &= \frac{\cos(\lambda x_2)}{2\pi} \int_{x_1-t}^{x_1+t} \int_{-\pi/2}^{\pi/2} \cos(\lambda \sqrt{t^2 - (x_1-y_1)^2} \sin z) dz dy_1 \\ &= \frac{1}{2} \int_{x_1-t}^{x_1+t} J_0(\lambda \sqrt{t^2 - (x_1-y_1)^2}) \psi(y_1)\, dy_1 \ . \end{aligned}$$