SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #2 Solutions

Problem 1. Suppose that $\Sigma \subset \mathbb{R}^d$ is a regular hypersurface, that is there exists an open set $U \subset \mathbb{R}^{d-1}$ and an injective function $x \in C^1(U, \mathbb{R}^d)$ such that $x(U) = \Sigma$ and that the rank of the Jacobian matrix Dx is equal to $d-1$ for all $u \in U$. The pair (U, x) is a *parametrization* of Σ.

Suppose that (V, y) is another parametrization of Σ. Prove that for all $f \in C(\Sigma)$ we have

$$\int_U f(x(u))\sqrt{\det[Dx(u)^T Dx(u)]}\, du = \int_V f(y(v))\sqrt{\det[Dy(v)^T Dy(v)]}\, dv\ .$$

Note that this identity shows that the definition of the surface integral $\int_\Sigma f\, dS$ given in the lecture is independent of the parametrization used.

Proof. Define $g = y^{-1} \circ x$. Then $g\ U \to V$ is a diffeomorphism of class C^1. Using the transformation formula for multiple integrals

$$\int_{g(U)} h(v)\, dv = \int_U h(g(u))|\det Dg(u)|\, du$$

gives with $h(v) = f(y(v))\sqrt{\det[Dy(v)^T Dy(v)]}$ since by the chain rule $Dx = Dy\, Dg$

$$\begin{aligned}
\int_V f(y(v))\sqrt{\det[Dy(v)^T Dy(v)]}\, dv &= \int_U f(x(u))\sqrt{[Dx\, Dg^{-1}]^T\, Dx\, Dg^{-1}}\, |\det Dg|\, du \\
&= \int_U f(x(u))\sqrt{Dg^{-T}\, Dx^T Dx\, Dg^{-1}}\, |\det Dg(u)|\, du \\
&= \int_U f(x(u))\sqrt{Dx^T Dx}\ \sqrt{Dg^{-1}}\sqrt{Dg^{-1}}\, |\det Dg(u)|\, du \\
&= \int_U f(x(u))\sqrt{Dx(u)^T Dx(u)}\, du\ ,
\end{aligned}$$

where the product rule of the determinant was used. □

The fact that g is a diffeomorphm is not obvious. This follows from the fact that Σ is a regular surface and can be proved as in Proposition 3.1.9 in [1].

Problem 2. Prove the converse of Theorem 2.2: Suppose that $\Omega \subset \mathbb{R}^d$ open, $u \in C^2(\Omega, \mathbb{R})$, and that

$$u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\, dS(y)$$

for all $B_r(x) \subset \Omega$. Then $\Delta u = 0$ in Ω.

Proof. Fix x and $r > 0$, then for all $0 < s < r$, the function

$$\Phi(s) = \frac{1}{|\partial B_s(x)|} \int_{\partial B_s(x)} u(y)\, dS(y)$$

is constant. On the other hand, as in the proof of Theorem 2.2 in the lecture, one computes

$$\Phi'(s) = |\partial B_s(x)| \int_{B_s(x)} \Delta u(y)\, dy$$

Hence, the integral of Δu over any ball inside of Ω vanishes. Consequently $\Delta u = 0$ in Ω. □

Problem 3. Suppose that $a \in \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, and that $f \in C^1(\mathbb{R}^2)$. Use the method of characteristics to find a solution to the initial value problem

$$\begin{cases} \partial_t u + a\partial_x u = f(t,x) & (t,x) \in \mathbb{R}^2 \ , \\ u(0,x) = u_0(x) & x \in \mathbb{R} \ . \end{cases}$$

Let $\gamma(t) = (t, at + \underline{x})$ be the characteristic line through the point $(0, \underline{x})$. Along this line a solution u to the differential equation above satisfies the ordinary differential equation

$$\frac{d}{dt} u(y(t)) = f(\gamma(t)) \ .$$

Integration over the time interval $(0,t)$ gives

$$u(t, at + \underline{x}) - u(0, \underline{x}) = \int_0^t f(s, as + \underline{x})\, ds \ .$$

Now set $x = at + \underline{x}$. Then

$$u(t,x) - u(0, x - at) = \int_0^t f(s, x - a(t-s))\, ds$$

and using the initial condition one obtains

$$u(t,x) = u_0(x - at) + \int_0^t f(s, x - a(t-s))\, ds \ .$$

References

[1] Christian Bär. *Elementary differential geometry*. Cambridge University Press, Cambridge, 2010. Translated from the 2001 German original by P. Meerkamp.