SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #3 Solutions

Problem 1. Let $x \in \mathbb{R}^d$ be a column vector and let I be the $d \times d$ identity matrix. Prove that $\det(I + xx^T) = 1 + |x|^2$.

Proof. Note that x is an eigenvector of the matrix $I + xx^T$ with corresponding eigenvalue $1 + |x|^2$:

$$(I + xx^T)x = x + x(x^T x) = (1 + |x|^2)x$$

Any vector $y \in \mathbb{R}^d$ perpendicular to x, that is $x^T y = 0$ is also an eigenvector with eigenvalue one:

$$(I + xx^T)y = y$$

The eigenspace corresponding to this eigenvalue is $d - 1$ dimensional. Recall that the determinant of a square matrix is equal to the product of its eigenvalues, counting multiplicities. Thus

$$\det(I + xx^T) = 1 + |x|^2 \ .$$

□

Problem 2. Compute the distributional derivative $\frac{d^4}{dx^4}|x|^3$.

Solution. By the definition of the distributional derivative

$$\frac{d^4}{dx^4}|x|^3(\varphi) = |x|^3\left(\frac{d^4\varphi}{\partial x^4}\right) \ .$$

The expression on the right-hand side can be expanded. This leads to

$$\begin{aligned}|x|^3\left(\frac{d^4\varphi}{\partial x^4}\right) &= \int_0^\infty x^3\frac{d^4\varphi}{\partial x^4}(x)\,dx - \int_{-\infty}^0 x^3\frac{d^4\varphi}{\partial x^4}\,dx = -\int_0^\infty 3x^2\frac{d^3\varphi}{\partial x^3}\,dx + \int_{-\infty}^0 3x^2\frac{d^3\varphi}{\partial x^3} \\ &= \int_0^\infty 6x\frac{d^2\varphi}{\partial x^2}\,dx - \int_{-\infty}^0 6x\frac{d^2\varphi}{\partial x^2}\,dx = -6\int_0^\infty \frac{d\varphi}{\partial x}\,dx + 6\int_{-\infty}^0 \frac{d\varphi}{\partial x}\,dx = 12\varphi(0)\end{aligned}$$

Hence we have established the formula

$$\frac{d^4}{dx^4}|x|^3 = 12\delta_0$$

in the sense of distributions.

Problem 3. Given $f \in L_1(\mathbb{R})$ and $a \in \mathbb{R}$. Show that the function $u(t,x) = f(x - at)$ is a distributional solution to the PDE $u_t + au_x = 0$, that is

$$u_t(\varphi) + au_x(\varphi) = 0 \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^2) \ .$$

Solution. Note that $u(t,x)$ defines a distribution since for $\varphi \in C_0^\infty(\mathbb{R}^2)$ since using the substitution $y = x - at$ and Fubini's Theorem one has

$$\begin{aligned} u(\varphi) &= \int_{\mathbb{R}^2} u(t,x)\varphi(t,x)\,dxdt \\ &= \int_{\mathbb{R}^2} f(y)\varphi(t,y+at)\,dydt = \int_{-\infty}^{\infty} f(y)\left\{\int_{-\infty}^{\infty}\varphi(t,y+at)dt\right\}dy\ . \end{aligned}$$

Note that the last integral is finite since $\varphi \in \mathscr{D}(\mathbb{R}^2)$ and $f \in L_1(\mathbb{R})$. By the transformation formula for multiple integrals we conclude that $u\varphi \in L_1(\mathbb{R}^2)$. However, it may be worth noting that u is usually not in $L_1(\mathbb{R}^2)$. For example, choose $f = \chi_{(0,1)} \in L_1(\mathbb{R})$ and observe that

$$\int_{\mathbb{R}^2} \chi_{(0,1)}(x-at)\,dxdt = \int_{\mathbb{R}}\left\{\int_{at}^{at+1} dx\right\}dt = \int_{\mathbb{R}} dt = \infty\ .$$

Compute now, using the definition of the distributional derivative and the substitution $y = x - at$, $s = t$

$$\begin{aligned} u_t(\varphi) + au_x(\varphi) &= -\int_{\mathbb{R}^2} u(t,x)[\varphi_t(t,x) + a\varphi_x(t,x)]\,dxdt \\ &= -\int_{\mathbb{R}^2} f(y)[\varphi_t(s,y+as) + a\varphi_x(s,y+as)]\,dyds \\ &= -\int_{\mathbb{R}^2} f(y)\frac{d}{ds}\varphi(s,y+as)\,dyds = -\int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}}\frac{d}{ds}\varphi(s,y+as)\,ds\right)dy = 0\ , \end{aligned}$$

since the function φ has compact support. Just in case: The partial derivatives of φ in the second integral are

$$\varphi_t(s,y+as) = \left.\frac{\partial\varphi}{\partial t}\right|_{t=s,x=y+as}$$

and

$$\varphi_x(s,y+as) = \left.\frac{\partial\varphi}{\partial x}\right|_{t=s,x=y+as}\ .$$