**SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II**
**LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN**

**Homework #4** due 5/17

**Problem 1.** Suppose that $\Omega \subset \mathbb{R}^d$ is of class $C^1$. Fix $\underline{x} \in \partial\Omega$. Then there exists a neighborhood $\mathscr{U}(\underline{x})$ and a function $g \in C^1$ such that

$$\partial\Omega \cap \mathscr{U}(\underline{x}) = \{x \in \mathscr{U}(\underline{x}) \ : \ x_d = g(x')\} \quad \text{and} \quad \Omega \cap \mathscr{U}(\underline{x}) = \{x \in \mathscr{U}(\underline{x}) \ : \ x_d > g(x')\} \ .$$

Let $\chi_\Omega$ be the characteristic function of $\Omega$. Prove that

$$\partial_j \chi_\Omega = -\nu_j \, dS \qquad \text{for all } x \in \mathscr{U}(\underline{x})$$

(and hence for all $x \in \mathbb{R}^d$). Here $\nu$ denotes the the exterior unit normal vector field along $\partial\Omega$ and $dS$ denotes the surface measure on $\partial\Omega$. Hint: In $\mathscr{U}(\underline{x})$ we have

$$\chi_\Omega(x) = \lim_{\varepsilon \to 0} h\left(\frac{x_d - g(x')}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \ ,$$

pointwise and in the sense of distributions, where $h \in C^\infty(\mathbb{R}, [0,1])$ satisfies $h(t) = 1$ for all $t \in (1, \infty)$ and $h(t) = 0$ for all $t \in (-\infty, 0)$.

**Problem 2.** Use the result of problem above to give a proof of Theorem 3.3. (Gauss's Theorem).

**Problem 3.** a.) Given is a compact set $K \subset U$ where $U$ is an open set in $\mathbb{R}^d$. Use Theorem 3.9 to establish the existence of a function $\eta \in C_0^\infty(U)$ which satisfies $\eta(x) = 1$ for all $x \in K$.

b.) Use part a.) to establish the existence of a partition of unity $\{\eta_j\}_{j=1}^m$ subordinate to the finite open cover $\{U_j\}_{j=1}^m$ of $\overline{\Omega} \subset \mathbb{R}^d$, that is, a family of functions $\eta_j \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \eta_j \subset U_j$, $0 \leq \eta_j \leq 1$ for $j = 1, 2, ..., m$, and $\sum_{j=1}^m \eta_j = 1$ for all $x \in \Omega$. You may assume that $\Omega$ is a bounded set.