SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #4 Solutions

Problem 1. Suppose that $\Omega \subset \mathbb{R}^d$ is of class C^1. Fix $\underline{x} \in \partial\Omega$. Then there exists a neighborhood $\mathscr{U}(\underline{x})$ and a function $g \in C^1$ such that

$$\partial\Omega \cap \mathscr{U}(\underline{x}) = \{x \in \mathscr{U}(\underline{x}) \ : \ x_d = g(x')\} \quad \text{and} \quad \Omega \cap \mathscr{U}(\underline{x}) = \{x \in \mathscr{U}(\underline{x}) \ : \ x_d > g(x')\} \ .$$

Let χ_Ω be the characteristic function of Ω. Prove that

$$\partial_j \chi_\Omega = -\nu_j \, dS \qquad \text{for all } x \in \mathscr{U}(\underline{x})$$

(and hence for all $x \in \mathbb{R}^d$). Here ν denotes the the exterior unit normal vector field along $\partial\Omega$ and dS denotes the surface measure on $\partial\Omega$. Hint: In $\mathscr{U}(\underline{x})$ we have

$$\chi_\Omega(x) = \lim_{\varepsilon\to 0} h\left(\frac{x_d - g(x')}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \ ,$$

pointwise and in the sense of distributions, where $h \in C^\infty(\mathbb{R}, [0,1])$ satisfies $h(t) = 1$ for all $t \in (1, \infty)$ and $h(t) = 0$ for all $t \in (-\infty, 0)$.

Proof. Recall that

$$\nu(x') = \frac{(\nabla g(x'), -1)}{\sqrt{1 + |\nabla g(x')|^2}} \ .$$

Compute, using the chain rule

$$\partial_j h\left(\frac{x_d - g(x')}{\varepsilon}\right) = h'\left(\frac{x_d - g(x')}{\varepsilon}\right) \frac{\nu_j(x')\sqrt{1 + |\nabla g(x')|^2}}{\varepsilon} \ , \qquad \text{for } j = 1, ..., d \ .$$

The distributional derivative of χ_Ω can be computed as follows. Choose $\varphi \in C_0^\infty(\mathscr{U}(\underline{x}))$. Then, relying on Proposition 3.7 and using the change of variables $s = (x_d - g(x'))/\varepsilon$, one obtains

$$\begin{aligned}
\partial_j \chi(\varphi) &= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} h'\left(\frac{x_d - g(x')}{\varepsilon}\right) \frac{\nu_j(x')\sqrt{1 + |\nabla g(x')|^2}}{\varepsilon} \varphi(x)\, dx \\
&= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{d-1}} \frac{1}{\varepsilon} \left\{ \int_{\mathbb{R}} h'\left(\frac{x_d - g(x')}{\varepsilon}\right) \varphi(x', x_d) dx_d \right\} \nu_j(x')\sqrt{1 + |\nabla g(x')|^2}\, dx' \\
&= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{d-1}} \left\{ \int_0^1 h'(s)\varphi(x', \varepsilon s + g(x')) ds \right\} \nu_j(x')\sqrt{1 + |\nabla g(x')|^2}\, dx' \\
&= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{d-1}} \varphi(x', \varepsilon + g(x'))\nu_j(x')\sqrt{1 + |\nabla g(x')|^2}\, dx' \\
&\quad + \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{d-1}} \left\{ \int_0^1 h(s) \frac{d}{ds}\varphi(x', \varepsilon s + g(x')) ds \right\} dx' \\
&= -\int_{\mathbb{R}^{d-1}} \varphi(x', g(x'))\nu_j(x')\sqrt{1 + |\nabla g(x')|^2}\, dx' = -\int_{\partial\Omega} \varphi(x)\nu_j(x) dS(x)
\end{aligned}$$

This shows that

$$\partial_j \chi_\Omega = -\nu_j dS \quad \text{in } \mathscr{D}'(\mathscr{U}(\underline{x})) \,.$$

Of course, since the choice of $\underline{x}$ was arbitrary, one obtains

$$\partial_j \chi_\Omega = -\nu_j dS \quad \text{in } \mathscr{D}'(X)$$

where $X \subset \mathbb{R}^d$ is open and $X \cap \partial\Omega$ is the graph of a C^1-function. Using a partition of unity, one obtains

$$\partial_j \chi_\Omega = -\nu_j dS \quad \text{in } \mathscr{D}'(\mathbb{R}^d) \,.$$

□

Problem 2. Use the result of problem above to give a proof of Theorem 3.3. (Gauss's Theorem).

Proof. Let $f \in C^1(\mathbb{R}^d)$. Then using the solution to the problem above

$$\int_\Omega \partial_j f dx = \int_{\mathbb{R}^d} \chi_\Omega(x)\partial_j f(x)\, dx = -\partial_j\chi(f) = (\nu_j dS)(f) = \int_{\partial\Omega} f(x)\nu_j(x)\, dS(x) \,.$$

Technically, a distribution is a linear functional on a function in $\mathscr{D}(\mathbb{R})$. However, the distributions χ and dS are of order zero and can view as linear functionals on the continuous functions. □

Problem 3. a.) Given is a compact set $K \subset U$ where U is an open set in $\mathbb{R}^d$. Use Theorem 3.9 to establish the existence of a function $\eta \in C_0^\infty(U)$ which satisfies $\eta(x) = 1$ for all $x \in K$.

b.) Use part a.) to establish the existence of a partition of unity $\{\eta_j\}_{j=1}^m$ subordinate to the finite open cover $\{U_j\}_{j=1}^m$ of $\overline{\Omega} \subset \mathbb{R}^d$, that is, a family of functions $\eta_j \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \eta_j \subset U_j$, $0 \le \eta_j \le 1$ for $j = 1, 2, ..., m$, and $\sum_{j=1}^m \eta_j = 1$ for all $x \in \Omega$. You may assume that Ω is an open and bounded set and that each U_j is bounded, $j = 1, 2, ..., m$.

Solution. a.) There exists $\varepsilon > 0$ such that $K^{3\varepsilon} \subset U$ where $K^\varepsilon = \{x \in \mathbb{R}^d : \operatorname{dist}(\partial K, x) \le \varepsilon\}$. Now set $\eta = \psi_\varepsilon * \chi_{K^\varepsilon}$ where ψ_ε is the family of Friedrichs mollifiers defined in the lecture and χ_A is the characteristic function of $A \subset \mathbb{R}^d$. Note that for $x \in \mathbb{R}^d \setminus K^{2\varepsilon}$ we have

$$\eta(x) = \int_{\mathbb{R}^d} \psi_\varepsilon(y-x)\chi_{K^\varepsilon}(y)\, dy = \int_{K^\varepsilon} \psi_\varepsilon(y-x)\, dy = 0$$

since $\psi_\varepsilon(z) = 0$ for $|z| \ge \varepsilon$. This proves $\eta \in C_0^\infty(U)$. Furthermore, for $x \in K$ we have

$$\eta(x) = \int_{\mathbb{R}^d} \psi_\varepsilon(y-x)\chi_{K^\varepsilon}(y)\, dy = \int_{K^\varepsilon} \psi_\varepsilon(y-x)\, dy = \int_{\mathbb{R}^d} \psi_\varepsilon(y-x) dy = 1 \,,$$

since for $y \notin K^\varepsilon$ and we have $\psi_\varepsilon(y-x) = 0$.

b.) At first one constructs a new open cover V_j of $\overline{\Omega}$ with the property that $\overline{V}_j \subset U_j$. This is done as follows. Set

$$F_1 = \overline{\Omega} \setminus \bigcup_{j=2}^m U_j \subset U_1 \,.$$

Note that F_1 is a compact set. Hence, there exists an open set $V_1 \supset F_1$ such that $\overline{V}_1 \subset U_1$. Similarly one finds the open sets $V_2, ..., V_m$. Set

$$F_k = \overline{\Omega} \setminus \left[\bigcup_{j=1}^{k-1} V_j \cup \bigcup_{j=k+1}^{m} U_j \right] \subset U_k \ , \qquad \text{for } k = 2, ..., m \ .$$

and choose open sets $V_j \subset U_j$ such that $\overline{V}_j \subset U_j$ and $F_j \subset V_j$ for $j = 2, ..., m$. Note that the definition of F_k requires only the knowledge of $V_1, .., V_{k-1}$.

According to part a.) there exist functions ϕ_j such that $\phi \in C_0^\infty(U_j)$, $0 \leq \phi_j \leq 1$ for $j = 1, 2, .., m$, and $\phi(x) = 1$ for all $x \in \overline{V}_j$. Since $\overline{\Omega} \subset \bigcup_{k=1}^m V_k$, there exists an open set $V \supset \overline{\Omega}$ such that $V \subset \bigcup_{k=1}^m V_k$. Using part a.) there exists a function $\phi \in C_0^\infty(\bigcup_{k=1}^m V_k)$ with $0 \leq \phi \leq 1$ and $\phi(x) = 1$ for all $x \in V$. Finally, the functions ϕ_j and ϕ can be considered as functions in $C_0^\infty(\mathbb{R}^d)$.

Set now

$$\eta_j(x) = \begin{cases} \dfrac{\phi_j(x)}{(1 - \phi(x)) + \sum_{k=1}^{m} \phi_k(x)} & \text{for} \quad x \in U_j \ , \\ 0 & \text{for} \quad x \notin U_j \ , \end{cases} \qquad \text{for } j = 1, 2, ..., m \ .$$

and observe that $\eta_j \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \eta_j \leq 1$ for $j = 1, 2, .., m$, and that $\sum_{j=1}^{m} \eta_j(x) = 1$ for all $x \in \Omega$. The introduction of the function ϕ ensures that no division by zero occurs in this definition.

Note that this construction works - mutatis mutandis - in the situation where each U_j is a bounded, open subset of Ω such that $\overline{U}_j \subset \Omega$, every compact subset of Ω intersects only finitely many U_j and

$$\Omega = \bigcup_{j=1}^{\infty} U_j \ .$$

For details we refer to the book by Renardy and Rogers [1, Theorem 5.6]. This has been used in the proof of Theorem 3.13.

References

[1] Michael Renardy and Robert C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1993.