SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #5 due 5/24

Problem 1. Prove Corollary 3.15 from the lecture: If $v, w \in H^1(\Omega) = W_2^1(\Omega)$, then $u = vw \in W_1^1(\Omega)$ and

$$\partial_j u = (\partial_j v)w + v(\partial_j w) \qquad \text{for } j = 1, 2, ..., d\,.$$

Problem 2. Note that the operator S defined in Proposition 3.17 is a continuous linear functional on the Hilbert space $H^1(\mathbb{R})$. According to the Riesz representation theorem from functional analysis, $H^1(\mathbb{R})$ can be identified with its own dual space. More precisely, if S is a continuous linear functional there exists a function $v \in H^1(\mathbb{R})$ such that

$$Sf = \int_{\mathbb{R}} f'(x)v'(x)\,dx + \int_{\mathbb{R}} f(x)v(x)\,dx$$

Find the function $v \in H^1(\mathbb{R})$ which corresponds to the linear functional S.

Problem 3. In analogy to the definition of the weak solution to the Dirichlet problem (Definition 3.21), define the weak solution $u \in H^1(\Omega)$ to the Neumann problem

$$-\Delta u = f \in L_2(\Omega)\,,$$
$$\partial_\nu u\Big|_{\partial\Omega} = g \in L_2(\partial\Omega)\,,$$

and prove that a weak solution u to the Neumann problem with regularity $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution. Here $\partial_\nu u = \nu \cdot \nabla u$ is the directional derivative in direction of the exterior unit normal ν along $\partial\Omega$. Can you tell why the homogeneous Neumann boundary condition is referred to as the *natural boundary condition* ?