SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #5 Solutions

Problem 1. Prove Corollary 3.15 from the lecture: If $v, w \in H^1(\Omega) = W_2^1(\Omega)$, then $u = vw \in W_1^1(\Omega)$ and

$$\partial_j u = (\partial_j v)w + v(\partial_j w) \qquad \text{for } j = 1, 2, ..., d\ .$$

Proof. With $v, w \in H^1(\Omega)$ we know that $v, w \in L_2(\Omega)$ and $\partial_j v, \partial_j w \in L_2(\Omega)$ for $j = 1, 2, ..., d$. By Hölder's ineqality, we note that

$$\int_\Omega |u|\, dx = \int_\Omega |vw|\, dx \leq \|v\|_{L_2(\Omega)} \|w\|_{L_2(\Omega)}\ .$$

This proves $u \in L_1(\Omega)$. By Theorem 3.13 we know that there exists sequences $v_n, w_n \in C^\infty(\Omega) \cap H^1(\Omega)$ such that $\|v_n - v\|_{H^1(\Omega)} \to 0$ and $\|w_n - w\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. Since the functions v_n, w_n are smooth we know that

$$\partial_j(v_n w_n) = (\partial_j v_n) w_n + v_n (\partial_j w_n)$$

We will show that $v_n w_n \to u$ in $W_1^1(\Omega)$. Observe that, using the triangle inequality and Hölder's inequality

$$\begin{aligned}\int_\Omega |u - v_n w_n|\, dx = \int_\Omega |vw - v_n w_n|\, dx &\leq \int_\Omega |v - v_n|\, |w|\, dx + \int_\Omega |w - w_n|\, |v_n|\, dx \\ &\leq \|v - v_n\|_{L_2(\Omega)} \|w\|_{L_2(\Omega)} + \|w - w_n\|_{L_2(\Omega)} \|v_n\|_{L_2(\Omega)} \longrightarrow 0\ , \qquad \text{as } n \to \infty\ .\end{aligned}$$

Furthermore, for $j = 1, 2, ..., d$ we get using the same idea

$$\begin{aligned}\int_\Omega |\partial_j u_n - (\partial_j v)w - v(\partial_j w)|\, dx &\leq \int_\Omega |(\partial_j v_n) w_n - (\partial_j v) w|\, dx + \int_\Omega |v_n(\partial_j w_n) - v(\partial_j w)|\, dx \\ &\leq \int_\Omega |\partial_j (v_n - v)|\, |w_n|\, dx + \int_\Omega |\partial_j v|\, |w - w_n|\, dx \\ &\quad + \int_\Omega |\partial_j (w_n - w)|\, |v_n|\, dx + \int_\Omega |\partial_j w|\, |v - v_n|\, dx \\ &\leq \|\partial_j (v_n - v)\|_{L_2(\Omega)} \|w_n\|_{L_2(\Omega)} + \|\partial_j v\|_{L_2(\Omega)} \|w - w_n\|_{L_2(\Omega)} \\ &\quad + \|\partial_j (w_n - w)\|_{L_2(\Omega)} \|v_n\|_{L_2(\Omega)} + \|\partial_j w\|_{L_2(\Omega)} \|v - v_n\|_{L_2(\Omega)} \longrightarrow 0 \text{ as } n \to \infty\ .\end{aligned}$$

This proves that $\partial_j u_n$ converges in L_1 to some limit function U_j. Observe now that for $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega u_n \partial_j \varphi\, dx = -\int_\Omega \partial_j u_n \varphi\, dx \longrightarrow -\int_\Omega U_j \varphi\, dx \qquad \text{as } n \to \infty\ .$$

Hence, the limit function U_j is the distributional derivative of u with respect to the x_j. $\square$

Problem 2. Note that the operator S defined in Proposition 3.17 is a continuous linear functional on the Hilbert space $H^1(\mathbb{R})$. According to the Riesz representation theorem from functional analysis, $H^1(\mathbb{R})$ can be identified with its own dual space. More precisely, if S is a continuous linear functional there exists a function $v \in H^1(\mathbb{R})$ such that

$$Sf = \int_{\mathbb{R}} f'(x)v'(x)\,dx + \int_{\mathbb{R}} f(x)v(x)\,dx$$

Find the function $v \in H^1(\mathbb{R})$ which corresponds to the linear functional S.

Solution. Choose $v(x) = e^{-|x|}/2$. Note that $v \in L_2(\mathbb{R})$ and that for $\phi \in C_0^\infty(\mathbb{R})$. Compute

$$\begin{aligned} v'(\varphi) &= -\frac{1}{2}\int_{\mathbb{R}} v(x)\phi'(x)\,dx = -\frac{1}{2}\int_{-\infty}^{0} e^{x}\phi'(x)\,dx - \frac{1}{2}\int_{0}^{\infty} e^{-x}\phi'(x)\,dx \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{x}\phi(x)\,dx - \frac{1}{2}\phi(0) - \frac{1}{2}\int_{0}^{\infty} e^{-x}\phi(x) + \frac{1}{2}\phi(0) \end{aligned}$$

which proves that the distributional derivative of v is the function

$$v'(x) = \begin{cases} \dfrac{e^{x}}{2} & \text{for } x < 0\ , \\ -\dfrac{e^{-x}}{2} & \text{for } x \geq 0\ . \end{cases}$$

One observe that $v'(x) \in L_2(\mathbb{R})$ and hence $v \in H^1(\mathbb{R})$.

Choose now $u \in C^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Then, using the results of the computation above

$$\begin{aligned} &\int_{\mathbb{R}} u(x)v(x)\,dx + \int_{\mathbb{R}} u'(x)v'(x)\,dx \\ &\qquad = \frac{1}{2}\int_{0}^{\infty} e^{-|x|}u(x)\,dx + \frac{1}{2}\int_{-\infty}^{0} e^{x}u'(x)\,dx - \frac{1}{2}\int_{0}^{\infty} u'(x)e^{-x}\,dx \\ &\qquad\qquad = \frac{1}{2}\int_{0}^{\infty} e^{-|x|}u(x)\,dx - \frac{1}{2}\int_{0}^{\infty} e^{-|x|}u(x)\,dx + u(0) = u(0) \end{aligned}$$

Problem 3. In analogy to the definition of the weak solution to the Dirichlet problem (Definition 3.21), define the weak solution $u \in H^1(\Omega)$ to the Neumann problem

$$\begin{aligned} -\Delta u &= f \in L_2(\Omega)\ , \\ \partial_\nu u\Big|_{\partial\Omega} &= g \in L_2(\partial\Omega)\ , \end{aligned}$$

and prove that a weak solution u to the Neumann problem with regularity $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution. Here $\partial_\nu u = \nu \cdot \nabla u$ is the directional derivative in direction of the exterior unit normal ν along $\partial\Omega$. Can you tell why the homogeneous Neumann boundary condition is referred to as the *natural boundary condition* ?

Solution. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution to the Neumann Problem. Multiplying the equation with some function $\phi \in C^\infty(\overline{\Omega})$ and integrate over Ω gives

$$-\int_{\Omega} (\Delta u)\phi\,dx = \int_{\Omega} f\phi\,dx\ .$$

Performing integration by parts in the left integral gives

$$\int_{\Omega} \nabla u \cdot \nabla\phi\,dx = \int_{\partial\Omega} \partial_\nu u\phi\,dS + \int_{\Omega} f\phi\,dx$$

and using the boundary condition results in

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_{\partial\Omega} g\phi \, dS + \int_\Omega f\phi \, dx \, .$$

Here the assumption that Ω is of class C^1 (or that the boundary is locally the graph of a Lipschitz function) is important. This justifies the following definition.

Definition. *Given $f \in L_2(\Omega)$ and $g \in L_2(\partial\Omega)$, a weak solution $u \in H^1(\Omega)$ to the Neumann problem satisfies the integral identity*

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_{\partial\Omega} g\phi \, dS + \int_\Omega f\phi \, dx \, .$$

for all $\phi \in H^1(\Omega)$.

Suppose now that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a weak solution to the Neumann problem. Then the identity above holds for all $u \in H^1(\Omega)$. Choosing $\phi \in C^1(\overline{\Omega})$ with $\phi\Big|_{\partial\Omega} = 0$ gives

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f\phi \, dx \, .$$

and integration by parts on the left-hand side results in

$$-\int_\Omega \Delta u \phi \, dx = \int_\Omega f\phi \, dx \, .$$

Since this identity holds for all $\phi \in C^1(\overline{\Omega})$ with zero boundary values, one obtains that $-\Delta u = f$ in Ω in the sense of distributions. If $f \in C(\overline{\Omega})$ the equality holds pointwise.

Next we will verify the boundary condition. We start again with the identity given in the definition above and choose $\phi \in C^1(\overline{\Omega})$. After integration by parts in the left integral one obtains

$$\int_{\partial\Omega} \partial_\nu u \phi \, dS - \int_\Omega \Delta u \phi \, dx = \int_{\partial\Omega} g\phi \, dS + \int_\Omega f\phi \, dx \, .$$

The volume integrals cancel each other out and one gets

$$\int_{\partial\Omega} \partial_\nu u \phi \, dS = \int_{\partial\Omega} g\phi \, dS \, .$$

Since this is true for all $\phi \in C(\partial\Omega)$, one has $\partial_\nu u = g$ in the sense of distributions. Again, if $g \in C(\overline{\Omega})$, then the equality holds pointwise.

Finally, note that if $g = 0$, the weak solution for the Neumann problem looks like

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f\phi \, dx \, .$$

for all $\phi \in H^1(\Omega)$. There is no boundary integral in this identity. This is the reason that the homogeneous Neumann condition is the natural boundary condition.