SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #6 due 5/31

Problem 1. Consider the Neumann problem for the Laplace operator in a bounded domain of class C^1. Define the Green function $N(x,y)$ (also sometimes called the Neumann function) and give a formal solution formula for Neumann problem

$$-\Delta u = f \in L_2(\Omega)\ ,$$
$$\partial_\nu u\Big|_{\partial\Omega} = g \in L_2(\partial\Omega)\ .$$

Problem 2. Derive the Green function $G(x,y)$ for the Dirichlet problem for the Laplacian in the ball $B_R(0)$ ($R>0$ fixed) in the case $d \geq 3$ and derive a formula for the solution of the Dirichlet problem

$$-\Delta u = 0 \text{ in } B_R(0)\ , \quad u = g \in C(\partial B_R(0))\ .$$

State a Theorem similar to Theorem 4.7 and prove it. Hint: For $x \neq 0$ set

$$G(x,y) = \frac{1}{d(d-2)\omega_d}\left[|x-y|^{2-d} - a|bx-y|^{2-d}\right]$$

and choose $a \in \mathbb{R}$ and $b > R/|x|$ such that $G(x,y) = 0$ for all $0 \neq x \in B_R(0)$ and $y \in \partial B_R(0)$.

Problem 3. *The chain rule for Sobolev functions.* Suppose that $f \in C^1(\mathbb{R}) \cap W^1_\infty(\mathbb{R})$ and let $u \in W^1_p(\Omega)$ for some $p \in [1,\infty)$ where $\Omega \subset \mathbb{R}^d$ is open and bounded. Prove that $f \circ u \in W^1_p(\Omega)$ and $\partial_j(f\circ u) = f'(u)\partial_j u$ in $L_p(\Omega)$ for $j = 1,..,d$. Does this chain rule also hold in the case that $p = \infty$?