SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #6 Solutions

Problem 1. Consider the Neumann problem for the Laplace operator in a bounded domain of class C^1. Define the Green function $N(x, y)$ (also sometimes called the Neumann function) and give a formal solution formula for Neumann problem

$$-\Delta u = f \in L_2(\Omega) \ ,$$
$$\partial_\nu u\Big|_{\partial\Omega} = g \in L_2(\partial\Omega) \ .$$

If $\Psi(x)$ is a fundamental solution for the Laplace operator, and $u \in C^2(\overline{\Omega})$, then according to Proposition 4.5 we have with $G(x, y) = \Psi(x - y)$ that

$$u(x) = -\int_\Omega G(x, y)\Delta u(y)\, dy - \int_{\partial\Omega} \partial_{\nu(y)} G(x, y)u(y)\, dS(y) + \int_{\partial\Omega} G(x, y)\partial_{\nu(y)} u(y)\, dS(y) \ .$$

In the case of the Neumann problem we do not have any information about the function u on the boundary whereas Δu in Ω and $\nu \cdot \nabla u$ on $\partial\Omega$ are know. Hence, the Neumann function $N(x, y)$ is a fundamental solution which satisfies the condition that

$$\nu(y) \cdot \nabla_y N(x, y) = 0 \qquad \text{for all } x \in \Omega, y \in \partial\Omega \ .$$

Note that if this condition is satisfies, the formula above reads as

$$u(x) = -\int_\Omega N(x, y) f(y)\, dy + \int_{\partial\Omega} N(x, y) g(y)\, dS(y) \ .$$

Construction of the Neumann function. Let Φ be the fundamental solution introduced in Definition 4.1. Suppose that the function $H(x, y)$ satisfies the following Neumann boundary value problem

$$\Delta_y H(x, y) = 0 \qquad \text{for all } x, y \in \Omega,$$
$$\partial_{\nu(y)} H(x, y) = -\partial_{\nu(y)} \Phi(x - y) \quad \text{for all } y \in \partial\Omega, x \in \Omega \ .$$

Then the Neumann function is given by $N(x, y) = \Phi(x - y) + H(x, y)$.

Problem 2. Derive the Green function $G(x, y)$ for the Dirichlet problem for the Laplacian in the ball $B_R(0)$ ($R > 0$ fixed) in the case $d \geq 3$ and derive a formula for the solution of the Dirichlet problem

$$-\Delta u = 0 \text{ in } B_R(0) \quad u = g \in C(\partial B_R(0)) \ .$$

State a Theorem similar to Theorem 4.8 and prove it. Hint: For $x \neq 0$ set

$$G(x, y) = \frac{1}{d(d-2)\omega_d} \left[|x - y|^{2-d} - a|bx - y|^{2-d}\right]$$

and choose $a \in \mathbb{R}$ and $b > R/|x|$ such that $G(x, y) = 0$ for all $0 \neq x \in B_R(0)$ and $y \in \partial B_R(0)$.

Solution. Note that for $x \in B_r(0)$ the function $G(x,y)$ satisfies $-\Delta_y G(x,y) = \delta_x$ as long as $bx \notin B_R(0)$. Fix $x \in B_R(0) \setminus \{0\}$. Then $G(x,y) = 0$ holds if for all $|y| = R$

$$a^{2/(2-d)}[b^2|x|^2 - 2bx \cdot y + R^2] = |x|^2 - 2x \cdot y + R^2 .$$

Choose now $a^{(2-d)/2} = b$. Then the equation is true if and only if

$$b^2|x|^2 + R^2 = b(|x|^2 + R^2) \quad \text{or} \quad b(b-1)|x|^2 = (b-1)R^2 .$$

Hence, $b = R^2/|x|^2$ and the Green function for the ball is

$$\begin{aligned} G(x,y) &= \frac{1}{d(d-2)\omega_d}\left[|x-y|^{2-d} - \left(\frac{|x|}{R}\right)^{2-d}\left|\frac{R^2}{|x|^2}x - y\right|^{2-d}\right] \\ &= \frac{1}{d(d-2)\omega_d}\left[|x-y|^{2-d} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-d}\right] \\ &= \frac{1}{d(d-2)\omega_d}\left[|x-y|^{2-d} - (R^2 - 2x\cdot y + |x|^2|y|^2/R^2)^{2-d}\right] , \end{aligned}$$

which is well-defined also for $x = 0$.

Compute now, for $x \in B_R(0)$ and $|y| = R$, using the formula for the gradient of the fundamental solution given before Theorem 4.2,

$$\begin{aligned} \nu(y)\cdot\nabla_y G(x,y) &= \frac{1}{d\omega_d}\left[\frac{y}{R}\frac{x-y}{|x-y|^d} - \frac{y}{R}\left(\frac{|x|}{R}\right)^{2-d}\frac{R^2x/|x|^2 - y}{|R^2x/|x|^2 - y|^d}\right] \\ &= \frac{1}{d\omega_d}\left[\frac{y\cdot x - R^2}{R|x-y|^d} - \frac{|x|^d}{R^{3-d}}\frac{R^2x\cdot y - R^2|x|^2}{|R^2x - y|x|^2|^d}\right] \\ &= \frac{1}{d\omega_d}\left[\frac{y\cdot x - R^2}{R|x-y|^d} - \frac{1}{R}\frac{x\cdot y - |x|^2}{|x-y|^d}\right] = \frac{1}{d\omega_d R}\frac{|x|^2 - R^2}{|x-y|^d} \end{aligned}$$

Theorem. *Suppose that $g \in C(\partial B_R(0))$. Then the function*

$$u(x) = \frac{R^2 - |x|^2}{d\omega_d R}\int_{\partial B_R(0)} \frac{g(y)}{|x-y|^d}dS(y)$$

has the regularity $u \in C^\infty(B_R(0)) \cap L_\infty(B_R(0))$, satisfies the equation $-\Delta u = 0$ in $B_R(0)$, and $u(x) = g(x)$ for all $x \in \partial B_R(0)$ in the sense that for all $\underline{x} \in \partial B_R(0)$

$$\lim_{x_l \to \underline{x}} u(x_l) = g(\underline{x})$$

for all sequences $\{x_l\} \subset B_R(0)$ converging to $\underline{x} \in \partial B_R(0)$.

Proof. For all $x \in B_R(0)$ one verifies that $\partial_j u(x)$ can be computed by the differentiation into the integral which can be done since $x \neq y$. This can be done for all derivatives of higher order as well and gives $u \in C^\infty(B_R(0))$. Next compute for $x \in B_R(0)$ and $|y| = R$, using the quotient rule of differential calculus,

$$\Delta_x \frac{|x|^2 - R^2}{|x-y|^d} = \nabla_x \cdot \nabla_x \frac{|x|^2 - R^2}{|x-y|^d} = \nabla_x \cdot \frac{2x|x-y|^d - d(|x|^2 - R^2)|x-y|^{d-2}(x-y)}{|x-y|^{2d}}$$

$$= \nabla_x \cdot \frac{2x}{|x-y|^d} - \nabla_x \cdot \frac{d(|x|^2 - R^2)(x-y)}{|x-y|^{d+2}}$$

$$= \frac{2d}{|x-y|^d} - \frac{2d(|x|^2 - x\cdot y)}{|x-y|^{d+2}} - \frac{2dx\cdot(x-y) - d^2(|x|^2 - R^2)}{|x-y|^{d+2}} + \frac{d(d+2)(|x|^2 - R^2)}{|x-y|^{d+2}}$$

$$= \frac{2d(x-y)^2}{|x-y|^{d+2}} - \frac{4d[|x|^2 - x\cdot y] - 2d(|x|^2 - R^2)}{|x-y|^{d+2}} = \frac{2d[|y|^2 - R^2]}{|x-y|^{d+2}} = 0\ .$$

This shows that $-\Delta u = 0$, pointwise in $B_R(0)$. Next we will show that $u \in L_\infty(\Omega)$. For that purpose set

$$K(x,y) = \frac{1}{d\omega_d R} \frac{R^2 - |x|^2}{|x-y|^d}$$

and observe that for $K > 0$ for $x \in B_R(0)$. Then, choosing $g \equiv 1$ and using Proposition 4.5 with $\Omega = B_R(0)$ we obtain

$$\int_{\partial B_R(0)} K(x,y)dS(y) = 1$$

From here the proof is completed as the proof of Theorem 4.7. ☐

A perhaps simpler way of proving $\Delta_x K(x,y) = 0$ for $x \in B_R(0)$ and $|y| = R$ is based on the fact that $G(x,y) = G(y,x)$ whenever $x \neq y$ and that $K(x,y) = \partial_{\nu(y)} G(x,y)$ for $y \in \partial B_R(0)$.

Problem 3. *The chain rule for Sobolev functions.* Suppose that $f \in C^1(\mathbb{R}) \cap W^1_\infty(\mathbb{R})$ and let $u \in W^1_p(\Omega)$ for some $p \in [1,\infty)$ where $\Omega \subset \mathbb{R}^d$ is open and bounded. Prove that $f \circ u \in W^1_p(\Omega)$ and $\partial_j(f\circ u) = f'(u)\partial_j u$ in $L_p(\Omega)$ for $j = 1,..,d$. Does this chain rule also hold in the case that $p = \infty$?

Proof. Fix $p \in [1,\infty)$. Note that $f\circ u \in L_p(\Omega)$ since $f \circ u \in L_\infty(\Omega)$ and Ω is bounded. By Theorem 3.13 there exists a sequence $u_m \in C^\infty(\Omega) \cap W^1_p(\Omega)$ such that $\|u_m - u\|_{W^1_p(\Omega)} \to 0$ as $m \to \infty$. Note that this convergence implies that $u_n \to u$ almost everywhere. The chain rule is true for differentiable functions, hence we have

$$\partial_j(f\circ u) = f'(u_m)\partial_j u_m \qquad \text{for } m = 1,2,... \text{ and } j = 1,2,...,d\ .$$

In what follows the positive integer $j \leq d$ is fixed. The sequence $f \circ u_m \in W^1_p(\Omega)$ converges to $f \circ u$ in $L_p(\Omega)$. Indeed, by the mean value theorem of differential calculus

$$\|f\circ u - f\circ u_m\|^p_{L_p(\Omega)} \leq \max_{\xi\in\mathbb{R}} |f'(\xi)| \int_\Omega |u(x) - u_m(x)|\, dx\ .$$

Convergence in $L_p(\Omega)$ implies convergence in $\mathscr{D}'(\Omega)$. By Proposition 3.7 we know that

$$\partial_j(f\circ u_m) \longrightarrow \partial_j(f\circ u) \quad \text{in } \mathscr{D}'(\Omega)\ .$$

Furthermore, we know that $f'(u)\partial_j u \in L_p(\Omega)$ since $f'(u) \in L_\infty(\Omega)$. Using the triangle inequality in $L_p(\Omega)$ we obtain

$$\begin{aligned}
&\left(\int_\Omega |f'(u_m)\partial_j u_m - f'(u)\partial_j u|^p \, dx\right)^{1/p} \\
&\qquad = \left(\int_\Omega \big|f'(u_m)\partial_j(u_m - u) + \big(f'(u_m) - f'(u)\big)\partial_j u\big|^p \, dx\right)^{1/p} \\
&\qquad \le \left(\int_\Omega |f'(u_m)\partial_j(u_m - u)|^p \, dx\right)^{1/p} + \left(\int_\Omega \big|\big(f'(u_m) - f'(u)\big)\partial_j u\big|^p \, dx\right)^{1/p} \\
&\qquad \le \|f'(u_m)\|_{L_\infty(\Omega)} \|u_m - u\|_{W_p^1(\Omega)} + \left(\int_\Omega \big|\big(f'(u_m) - f'(u)\big)\partial_j u\big|^p \, dx\right)^{1/p} \longrightarrow 0
\end{aligned}$$

as $m \to \infty$ where the continuity of f' and the Lebesgue dominated convergence theorem has been used. Here the observation that the continuity of f' implies that $f'(u_m) \to f'(u)$ almost everywhere in Ω is crucial. This shows that

$$f'(u_m)\partial_j u_m \longrightarrow f'(u)\partial_j u \qquad \text{in } L_p(\Omega)$$

and by the uniqueness of the limit on obtains

$$f'(u)\partial_j u = \partial_j(f \circ u) \in L_p(\Omega) \; .$$

□

If $u \in W_\infty^1(\Omega)$, then there does not need to exist a sequence of smooth functions converging to u in $W_\infty^1(\Omega)$. However, since Ω is bounded, we know that $u \in W_p^1(\Omega)$ for any $p \in [1, \infty)$ and we can use the statement proved above to state that $\partial_j(f \circ u) = f'(u)\partial_j u$ in $L_p(\Omega)$. However, the right-hand side is also in $L_\infty(\Omega)$ and that shows that the chain rule holds also for for compositions $f \circ u$ with $f \in C^1(\mathbb{R}) \cap W_\infty^1(\mathbb{R})$ and $u \in W_\infty^1(\Omega)$.

An interesting application of this chain rule pertains to problem 4.1 and results in a proof of the divergence theorem for bounded regions whose boundary is locally the graph of a Lipschitz function.