SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #7 Solutions

Problem 1. Prove that $C_0^\infty(\mathbb{R}^d)$ is dense in $H^k(\mathbb{R}^d)$ for all positive integers k and d.

Proof. At first consider a function $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ for all $|x| \le 1$ and $\chi \equiv 0$ for all $|x| \ge 2$. Given a function $u \in H^k(\mathbb{R}^d)$ the functions $u_m = \chi(x/m)u$ are in $H^k(\mathbb{R}^d)$ and are supported in $|x| < 2m$. One can show that $\|u - u_m\|_{H^k(\mathbb{R}^d)} \to 0$ as $m \to \infty$.

Now we will show that each $u \in H^k(\mathbb{R}^d)$ with compact support can be approximated by a sequence of functions in $C_0^\infty(\mathbb{R}^d)$. Suppose that $\psi \in C_0^\infty(B_1(0))$ is the function introduced at the beginning of subsection 3.2.2. extended by zero to all of $\mathbb{R}^d$. Let $\psi_n(x) = n^d\psi(nx)$. Given $u \in H^k(\mathbb{R}^d)$ with compact support, set $u_n = \psi_n * u$. Note that $u_n \in C_0^\infty(\mathbb{R}^d)$. Indeed, if u is supported in $B_R(0)$, then u_n is supported in $B_{R+1/n}$. We will show that $\|u_n - \psi_n * u\|_{H^k(\mathbb{R}^d)} \to 0$ as $n \to \infty$.

Set now $u_{nm} = \psi_n * u_m$. Then $u_{nn} \to u$ in $H^k(\mathbb{R}^d)$. □

Problem 2. Give a counterexample for the Poincaré inequality (Theorem 5.1) in $\mathring{H}^1(\mathbb{R}) = H^1(\mathbb{R})$. Construct a sequence $u_n \in C^\infty(\mathbb{R})$ of functions such that $\int_{\mathbb{R}} |u_n'(x)|^2\,dx \to 0$ while $\int_{\mathbb{R}} |u_n(x)|^2\,dx = 1$ for $n \to \infty$.

Consider

$$u_a(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}\,, \qquad a > 0\,.$$

Then one computes

$$\int_{\mathbb{R}} |u_a(x)|^2 = \sqrt{\frac{a}{\pi}} \int_{\mathbb{R}} e^{-ax^2}\,dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2}\,dy = 1$$

and

$$\begin{aligned}\int_{\mathbb{R}} |u_a'(x)|^2\,dx &= \sqrt{\frac{a}{\pi}} \int (-2ax)^2 e^{-ax^2}\,dx = 4a^2\sqrt{\frac{a}{\pi}} \int_{\mathbb{R}} x^2 e^{-ax^2}\,dx \\ &= \frac{4a}{\sqrt{\pi}} \int_{\mathbb{R}} y^2 e^{-y^2}\,dy = \frac{4a}{\sqrt{\pi}} \left[-\frac{y}{2}e^{-y^2}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int e^{-y^2}\,dy \right] = 2a\end{aligned}$$

Finally, observe that the sequence $u_{1/n}$ provides the counterexample.

Problem 3. Rellich selection theorem. *Suppose that Ω is a bounded, connected and open set of class C^1. Then every bounded sequence in $H^1(\Omega)$ has a strongly converging subsequence in $L_2(\Omega)$.*

Use this theorem to prove the Poincaré inequality $\|u\|_{L_2(\Omega)} \le C\|\nabla u\|_{L_2(\Omega)}$ for

a.) all $u \in \mathring{H}^1(\Omega)$.

b.) all $u \in H = \{u \in H^1(\Omega) \ :\ \int_\Omega u\,dx = 0\}$.

Hint: Argue by contradiction.

solution. We will focus on the second problem since the first one works exactly the same way. It is important that both spaces are complete. Suppose that the Poincaré inequality does not hold. Then there exists a sequence $u_n \in H$ such that

$$\|u_n\|_{L_2(\Omega)} \geq n\|\nabla u\|_{L_2(\Omega)} \ , \qquad n = 1, 2, ...$$

Note that the sequence $w_n = u_n / \|u_n\|_{L_2(\Omega)}$ satisfies

$$\|w_n\|_{L_2(\Omega)} = 1 \quad \text{and} \quad \|\nabla w_n\| \longrightarrow 0 \text{ as } n \to \infty \ . \tag{1}$$

Hence $\nabla w_n \to 0$ in $L_2(\Omega)$. Furthermore, by the Rellich selection theorem, there exists a subsequence of w_n (for convenience denoted by w_n) which converges strongly in $L_2(\Omega)$, i.e. $w_n \to w \in L_2(\Omega)$. The distributional derivative of w is computed using a function $\varphi \in C_0^\infty(\Omega)$. We have

$$\partial_j w(\varphi) = -\int_\Omega w \partial_j \varphi \, dx = -\lim_{n\to\infty} \int w_n \partial_j \varphi \, dx = \lim_{n\to\infty} \int \partial_j w_n \varphi \, dx = 0 \ , \qquad \text{for } j = 1, 2, ..., d \ ,$$

where the L_2 convergence of the gradient was used. Hence, $\nabla w = 0$ in $L_2(\Omega)$. This shows that $w \in H$ and w must be a constant. The only constant function in H is zero. On the other hand, formula (1) gives $\|w\|_{L_2(\Omega)} = 1$.