SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #8 due 06/14

Problem 1. Consider the Dirichlet problem for the Laplace equation in the unit disk, that is

$$-\Delta u = 0 \quad \text{in } B_1(0) \subset \mathbb{R}^2 , \qquad u = g \quad \text{in } \partial B_1(0) .$$

Suppose that g can be expanded into a Fourier series $g(\phi) = \sum_{k=0}^{\infty} a_k \cos(k\phi)$ and look for a solution of the form

$$u(r,\phi) = \sum_{k=0}^{\infty} b_k(r) \cos(k\phi) ,$$

where (r,ϕ) are polar coordinates.
a.) Show that the b_k have to satisfy the equation

$$\partial_r^2 b_k + \frac{1}{r}\partial_r b_k - \frac{k^2}{r^2} b_k = 0 \qquad \text{for } k = 1, 2, \dots .$$

Solve this differential equation. Since it is of second order, there are two linearly independent solutions. However, one has to be discarded. Why ?
b.) Give a condition on the sequence $\{a_k\}$ which guarantees that the energy $\int_{B_1(0)} |\nabla u|^2 \, dx$ of the series solution is finite.
c*.) Can you find a $g \in C(\partial B_1(0))$ such that the corresponding solution u does not have finite energy ?

Problem 2. Use the series expansion derived in Problem 1 to obtain the Poisson integral formula in $d = 2$. In higher dimensions this formula has been discussed in Homework 6, Problem 2.

Problem 3. Consider the function $f(x) = \chi_{[-a,a]} \in L_2(\mathbb{R})$.
a.) Compute the distributional derivative $f'(x)$ and show that $\Delta^h f \to 0$ almost everywhere.
b.) Show that the difference quotients $\Delta^h f$ do not converge to zero in $L_2(\mathbb{R})$.