SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #8 due 06/14

Problem 1. Consider the Dirichlet problem for the Laplace equation in the unit disk, that is

$$-\Delta u = 0 \quad \text{in } B_1(0) \subset \mathbb{R}^2 \,, \qquad u = g \quad \text{in } \partial B_1(0) \,.$$

Suppose that g can be expanded into a Fourier series $g(\phi) = \sum_{k=0}^{\infty} a_k \cos(k\phi)$ and look for a solution of the form

$$u(r,\phi) = \sum_{k=0}^{\infty} b_k(r)\cos(k\phi) \,,$$

where (r,ϕ) are polar coordinates.
a.) Show that the b_k have to satisfy the equation

$$b_k''(r) + \frac{1}{r}b_k'(r) - \frac{k^2}{r^2}b_k = 0 \qquad \text{for } k = 1, 2, \ldots \,.$$

Solve this differential equation. Since it is of second order, there are two linearly independent solutions. However, one has to be discarded. Why ?
Solution. Recall that the Laplacian in polar coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \,.$$

Differentiating the series $u(r,\phi)$ formally gives

$$\Delta u = \sum_{k=0}^{\infty} \left[b_k''(r) + \frac{1}{r}b_k'(r) - \frac{k^2}{r^2}b_k(r) \right] \cos(k\phi)$$

Since the functions $\cos(k\phi)$ are an orthogonal set in $L_2(0,2\pi)$, the series vanishes only if each b_k satisfies the differential equation

$$b_k''(r) + \frac{1}{r}b_k'(r) - \frac{k^2}{r^2}b_k(r) = 0 \,.$$

The general solutions of this equation are $b_0(r) = c_1 + c_2 \log r$ and $b_k(r) = c_1 r^k + c_2 r^{-k}$ for $k = 1, 2, \ldots$. In each case we set $c_2 = 0$ and discard the unbounded solution. We obtain

$$u(r,\phi) = \sum_{k=0}^{\infty} a_k r^k \cos(k\phi) \,. \tag{1}$$

Given that the series for g is convergent in the L_2-sense, that is $\sum_{k=0}^{\infty} a_k^2 < \infty$, the series for $u(r,\phi)$ is absolutely convergent for all $r < 1$.
b.) Give a condition on the sequence $\{a_k\}$ which guarantees that the energy $\int_{B_1(0)} |\nabla u|^2 \, dx$

of the series solution is finite.

Solution. Observe that

$$|\nabla u|^2 = |\partial_r u|^2 + \frac{1}{r^2}|\partial_\phi u|^2 \,.$$

Hence, using the orthogonality of the cosine and sine functions and $\int_0^{2\pi} \cos^2(k\phi)\,d\phi = \int_0^{2\pi} \sin^2(k\phi)\,d\phi = \pi$ for $k = 1, 2, ..,$ one obtains

$$\begin{aligned}\int_{B_1(0)} |\nabla u|^2\,dx &= \int_0^{2\pi}\int_0^1 \left[|\partial_r u|^2 + \frac{1}{r^2}|\partial_\phi u|^2\right] r\,dr d\phi \\ &= \pi\sum_{k=1}^\infty a_k^2 k^2 \int_0^1 r^{2k-1}\,dr + \pi\sum_{k=1}^\infty a_k^2 k^2 \int_0^1 r^{2k-1}\,dr = \pi\sum_{k=1}^\infty a_k^2 k \,.\end{aligned}$$

This integral is finite if the sum $\sum_{k=1}^\infty a_k^2 k < \infty$.

c*.) Can you find a $g \in C(\partial B_1(0))$ such that the corresponding solution u does not have finite energy ?

Problem 2. Use the series expansion derived in Problem 1 to obtain the Poisson integral formula in $d = 2$. In higher dimensions this formula has been discussed in Homework 6, Problem 2.

Solution. If

$$g(\phi) = \sum_{k=0}^\infty a_k \cos(k\phi)$$

then

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta \quad \text{and} \quad a_k = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(k\theta)\,d\phi \text{ for } k \geq 1 \,,$$

and

$$\int_0^{2\pi} g(\theta)\sin(k\theta) = 0 \quad \text{for } k = 1, 2 \,.$$

Moving these expression into formula one gives

$$\begin{aligned} u(r,\phi) &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta + \frac{1}{\pi}\sum_{k=1}^\infty r^k \left[\cos(k\phi)\int_0^{2\pi} g(\theta)\cos(k\theta)\,d\theta + \sin(k\phi)\int_0^{2\pi} g(\theta)\sin(k\theta)\,d\theta\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta + \frac{1}{\pi}\sum_{k=1}^\infty r^k \int_0^{2\pi} g(\theta)\cos(k(\theta-\phi))\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=1}^\infty r^k[e^{ik(\theta-\phi)} + e^{-ik(\theta-\phi)}]g(\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\left[\frac{1}{1-re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{1-re^{-i(\theta-\phi)}}\right]d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\frac{1-re^{-i(\theta-\phi)} + (1-re^{i(\theta-\phi)})re^{-i(\theta-\phi)}}{1-2r\cos(\theta-\phi)+r^2}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,d\theta \end{aligned}$$

Returning to Cartesian coordinates gives

$$u(x) = \frac{1-|x|^2}{2\pi} \int_S \frac{g(y)}{|x-y|^2}\, dS(y)\ .$$

which is the Poisson formula for the unit circle S.

Problem 3. Consider the function $f(x) = \chi_{[-a,a]} \in L_2(\mathbb{R})$.
a.) Compute the distributional derivative $f'(x)$ and show that $\Delta^h f \to 0$ almost everywhere.
Solution. Let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\frac{d}{dx} f(\varphi) = -\int_a^a \varphi'(x)\, dx = -\varphi(a) + \varphi(-a)$$

and hence $f' = \delta_{-a} + \delta_a$. Note that $f' = 0$ almost everywhere. For all $x \in \mathbb{R} \setminus \{a, -a\}$ we have

$$\Delta^h f(x) = \frac{f(x+h) - f(x)}{h} = 0$$

as soon as $|h| < \min\{|x-a|, |x+a|\}$. Hence, $\Delta^h f \to 0$ almost everywhere.
b.) Show that the difference quotients $\Delta^h f$ do not converge to zero in $L_2(\mathbb{R})$.
Solution. Suppose that $h > 0$. Then

$$\Delta^h f(x) = \begin{cases} 0 & \text{for} \quad x > a \\ -1/h & \text{for} \quad a-h < x \leq a \\ 0 & \text{for} \quad -a \leq x \leq a-h \\ 1/h & \text{for} \quad -a-h \leq x < -a \\ 0 & \text{for} \quad x < -a-h \end{cases}$$

and

$$\int_{\mathbb{R}} |\Delta^h f(x)|^2\, dx = \int_{a-h}^{a} \frac{1}{h^2}\, dx + \int_{-a-h}^{-a} \frac{1}{h^2}\, dx = \frac{2}{h}$$

which proves that $\Delta^h f$ does not converge to zero in $L_2(\mathbb{R})$. A similar computation can be done for $h < 0$.