SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #9 due 06/21

Problem 1. Suppose that $\Omega \subset \mathbb{R}^d$ is open and bounded. Use the Gagliardo-Nirenberg-Sobolev inequality (Theorem 6.3) to make the constant in the Poincaré inequality (Theorem 5.1) explicit. The constant does depend on d, p and $|\Omega|$.

Problem 2. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded region of class C^2 and let $f \in L_2(\Omega)$. A function $u \in \mathring{H}^2(\Omega)$ is a weak solution to the Dirichlet problem for the bi-Laplacian

$$\Delta^2 u = f \text{ in } \Omega \,, \qquad u = \partial_\nu u = 0 \text{ on } \partial\Omega \,,$$

if for all $v \in \mathring{H}^2(\Omega)$ is identity

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx$$

holds. Prove that for each $f \in L_2(\Omega)$ there exists a weak solution to the Dirichlet problem for the bi-Laplacian.

Problem 3. Recall that the Fourier transform $\mathscr{F}$ extends to a unitary operator on $L_2(\mathbb{R}^d)$, that is $\mathscr{F}'\mathscr{F} = \mathscr{F}\mathscr{F}' = I$. Here,

$$\mathscr{F}[f(x)](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx \quad \text{and} \quad \mathscr{F}'[g(\xi)](x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi) \, d\xi \,,$$

where $f, g \in \mathscr{S}(\mathbb{R}^d)$. Prove that $u \in H^k(\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi < \infty \,.$$