SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #9 due 06/21

Problem 1. Suppose that $\Omega \subset \mathbb{R}^d$ is open and bounded, $d \geq 2$. Use the Gagliardo-Nirenberg-Sobolev inequality (Theorem 6.3) to make the constant in the Poincaré inequality (Theorem 5.1) explicit. The constant does depend on d, p and $|\Omega|$.

First solution in the case $p \in [1, d)$. Using the definition of $\mathring{W}_p^1(\Omega)$ it will suffice to work with smooth, compactly supported functions. Recall the Gagliardo-Nirenberg-Sobolev inequality,

$$\|u\|_{L_{dp/(d-p)}(\Omega)} \leq \frac{(d-1)p}{(d-p)\sqrt{d}} \|\nabla u\|_{L_p(\Omega)} \qquad \text{for } 1 \leq p < d\,. \tag{1}$$

For $p < d$ observe that by Hölder's inequality

$$\|u\|_{L_p(\Omega)} \leq \left(\int |u|^{dp/(d-p)} \right)^{(d-p)/(dp)} \left(\int_\Omega dx \right)^{1/d} = |\Omega|^{1/d} \|u\|_{L_{dp/(d-p)}(\Omega)}\,, \tag{2}$$

since

$$\frac{d-p}{d} + \frac{p}{d} = 1\,.$$

Combining the two inequalities gives

$$\|u\|_{L_p(\Omega)} \leq |\Omega|^{1/d} \frac{(d-1)p}{(d-p)\sqrt{d}} \|\nabla u\|_{L_p(\Omega)}\,, \qquad \text{for } 1 \leq p < d\,.$$

Second solution for the case $p \in [d/(d-1), \infty)$. Given $p \in [1, \infty)$, choose q such that

$$p = \frac{dq}{d-q} \qquad \text{that is} \qquad q = \frac{pd}{d+p}\,.$$

Observe that $1 \leq q < d$; hence, the Gagliardo-Nirenberg-Sobolev inequality (1) is applicable with p replaced by q and gives

$$\|u\|_{L_p(\Omega)} \leq \frac{(d-1)q}{(d-q)\sqrt{d}} \|\nabla u\|_{L_q(\Omega)}\,.$$

Using Hölder's inequality as in (2) gives

$$\|\nabla u\|_{L_q(\Omega)} \leq \|\nabla u\|_{L_p(\Omega)} |\Omega|^{1/d}$$

and combining the last two inequalities gives

$$\|u\|_{L_p(\Omega)} \leq \frac{(d-1)q}{(d-q)\sqrt{d}} |\Omega|^{1/d} \|\nabla u\|_{L_p(\Omega)} = |\Omega|^{1/d} \frac{p(d-1)}{d\sqrt{d}} \|\nabla u\|_{L_p(\Omega)}\,,$$

since

$$\frac{(d-1)q}{d-q} = \frac{dq}{d-q} - \frac{q}{d-q} = p - \frac{p}{d} = \frac{p(d-1)}{d}\,.$$

Problem 2. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded region of class C^2 and let $f \in L_2(\Omega)$. A function $u \in \mathring{H}^2(\Omega)$ is a weak solution to the Dirichlet problem for the bi-Laplacian

$$\Delta^2 u = f \text{ in } \Omega , \qquad u = \partial_\nu u = 0 \text{ on } \partial\Omega ,$$

if for all $v \in \mathring{H}^2(\Omega)$ is identity

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx$$

holds. Prove that for each $f \in L_2(\Omega)$ there exists a weak solution to the Dirichlet problem for the bi-Laplacian.

Proof. We will apply the Lax-Milgram Lemma (Theorem 5.11) to the bilinear form

$$a(u, v) = \int_\Omega \Delta u \Delta v \, dx$$

on $H = \mathring{H}^2(\Omega)$. Using the Cauchy-Schwarz inequality one obtains the estimate

$$|a(u, v)| \le \|\Delta u\|_{L_2(\Omega)} \|\Delta v\|_{L_2(\Omega)} , \quad \text{for all } u, v \in \mathring{H}^2(\Omega) .$$

Furthermore,

$$a(u, u) = \|\Delta u\|^2_{L_2(\Omega)} , \quad \text{for all } u \in \mathring{H}^2(\Omega) .$$

In order to use these estimates for the required continuity and coercivity of a on $\mathring{H}^2(\Omega)$ we will introduce an equivalent norm on this Sobolev space. For that purpose we will use Theorem 5.18 which implies in our case ($L = -\Delta$, $g = 0$), that

$$\|u\|_{H^2(\Omega)} \le C \left\{ \|u\|_{L_2(\Omega)} + \|\Delta u\|_{L_2(\Omega)} \right\} \quad \text{for all } u \in \mathring{H}^2(\Omega) . \tag{3}$$

We will show that this estimate can be improved to

$$\|u\|_{H^2(\Omega)} \le C \|\Delta u\|_{L_2(\Omega)} \quad \text{for all } u \in \mathring{H}^2(\Omega) . \tag{4}$$

In other words, the expression $\|\Delta u\|_{L_2(\Omega)}$ is an equivalent norm on $\mathring{H}^2(\Omega)$. Assuming the validity of (4) gives the continuity and coercivity of the bilinear form a. Applying the Lax-Milgram lemma gives a unique solution to the variational equation $a(u, v) = \langle f, v \rangle$ for every $f \in H^{-2}(\Omega)$ which is the dual of $\mathring{H}^2(\Omega)$. Of course, every $f \in L_2(\Omega)$ is also a linear functional on $\mathring{H}^2(\Omega)$ via the formula

$$\langle f, v \rangle = \int_\Omega f v \, dx \quad \text{for all } v \in \mathring{H}^2(\Omega) .$$

It remains to prove (4). We argue by contradiction. If (4) is not correct, then there exists a sequence $u_n \in \mathring{H}^2(\Omega)$ such that

$$1 = \|u_n\|_{H^2(\Omega)} > n \|\Delta u_n\|_{L_2(\Omega)} \quad n = 1, 2, ...$$

A bounded sequence in $H^2(\Omega)$ is also bounded in $H^1(\Omega)$. Using the Rellich selection theorem (Corollary 6.8) provides a strongly converging subsequence (for convenience denoted again by u_n) in $L_2(\Omega)$, that is $\|u - u_n\|_{L_2(\Omega)} \to 0$ as $n \to \infty$. Applying estimate (3) to the difference $u_n - u_m$ shows that u_n is Cauchy in $H^2(\Omega)$ and hence, by the uniqueness of the limit one infers that $u \in \mathring{H}^2(\Omega)$. Because of $\Delta u_n \to 0$ in $L_2(\Omega)$ one knows also that $\Delta u = 0$. Using the maximum principle (Corollary 6.6) gives $u = 0$. However, this is a contradiction to $\|u_n\|_{H^2(\Omega)} = 1$ for all n. □

Problem 3. Recall that the Fourier transform $\mathscr{F}$ extends to a unitary operator on $L_2(\mathbb{R}^d)$, that is $\mathscr{F}'\mathscr{F} = \mathscr{F}\mathscr{F}' = I$. Here,

$$\mathscr{F}[f(x)](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx \quad \text{ and } \quad \mathscr{F}'[g(\xi)](x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\, d\xi \ ,$$

where $f, g \in \mathscr{S}(\mathbb{R}^d)$. Prove that $u \in H^k(\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi < \infty \ .$$

Proof. Suppose that $u \in H^k(\mathbb{R}^d)$. Then $\partial^\alpha u \in L_2(\mathbb{R}^d)$ for $|\alpha| \leq k$ if and only if $\widehat{\partial^\alpha u} = (i\xi)^\alpha \hat{u}(\xi)$. Hence, we conclude that $u \in H^k(\mathbb{R}^d)$ implies because of Parseval's identity that

$$\int_{\mathbb{R}^d} |\xi^{2\alpha}||\hat{u}(\xi)|^2 \, d\xi < \infty \qquad \text{for all } |\alpha| \leq k \ .$$

Using the multinomial theorem, in particular the formula,

$$(1 + |\xi|^2)^k = \sum_{|\alpha|=k} \frac{k!\xi^{2\alpha}}{\alpha!} \ ,$$

one can show that there exist constants $C_1, C_2 > 0$ such that

$$\sum_{|\alpha|\leq k} |\xi^{2\alpha}| \leq C(1 + |\xi|^2)^k \leq C_2 \sum_{|\alpha|\leq k} |\xi^{2\alpha}| \ .$$

The first inequality shows the implication $u \in H^k(\mathbb{R}^d)$ implies $\int_{\mathbb{R}^d}(1+|\xi|^2)^k|\hat{u}(\xi)|^2 \, d\xi < \infty$, the second inequality the opposite direction.

□