PDAEs with Uncertainties for the Analysis, Simulation and Optimization of Energy Networks

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Topic and background of the project

1. Analysis and simulation of PDE systems with stochastic algebraic constraints

Example: (gas) network model

$$\partial_t \varphi(v) + \partial_x \psi(v) + f(v) = 0$$
$$g(\mathcal{R}v, w) = Cu + CX_t$$

$v = v(x,t)$	pressures and flows for pipes
$w = w(t)$	pressures and flows for active elements (valves, ...)
$u = u(t)$	input/output flow/pressure
$\mathcal{R}$	boundary operator

fluctuations at input/output nodes modeled via mean-reverting Ornstein-Uhlenbeck process X_t

$$dX_t = \lambda(\mu - X_t)dt + \sigma dW_t, \qquad \lambda > 0$$

W_t Wiener process

2. Optimization problems with implicit probabilistic constraints

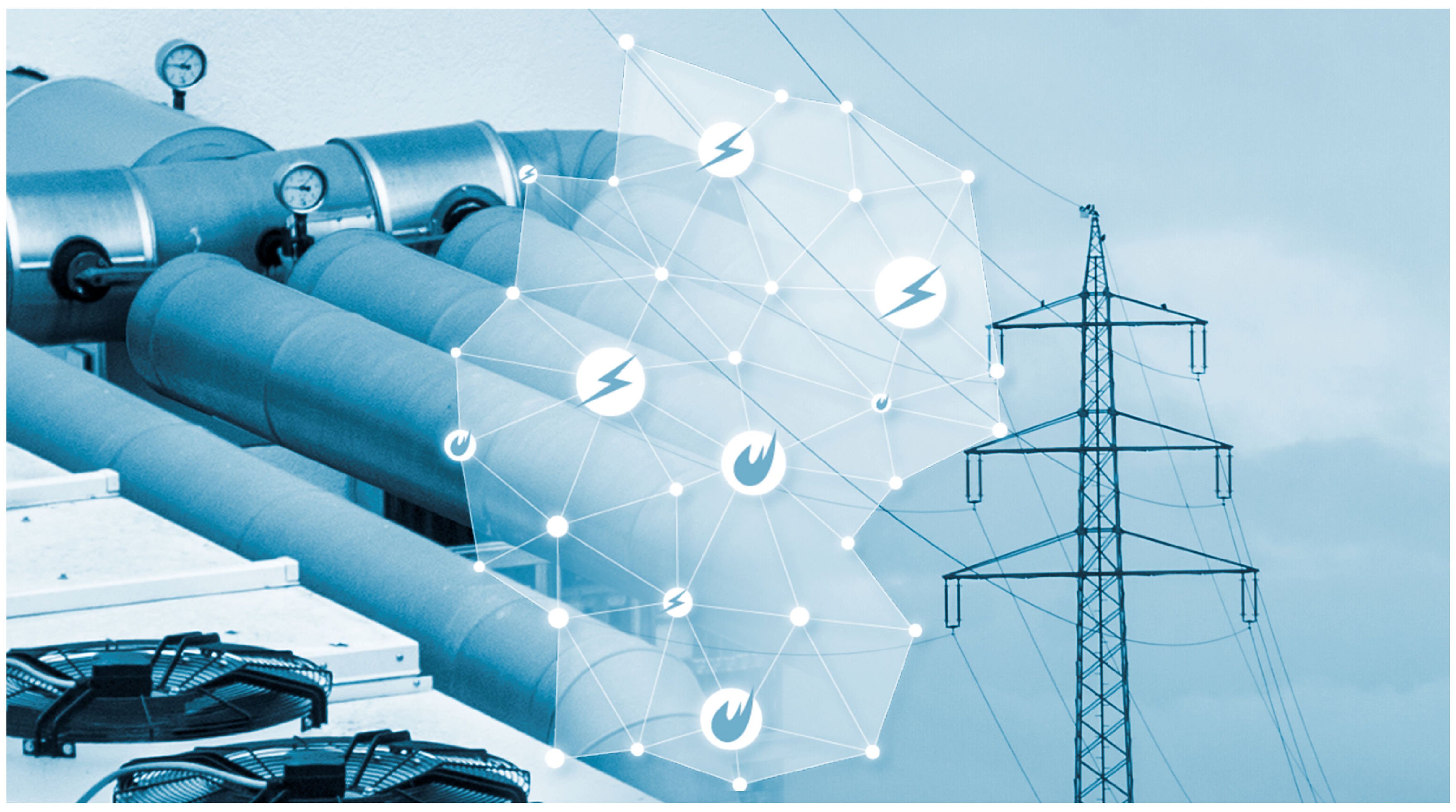
$$\min \left\{ f(u) \mid h(v'_\xi, v_\xi, \xi) = 0,\ \mathbb{P}[g(u, v_\xi) \leq 0] \geq p \right\} \quad (p \in (0,1))$$

Example: maximization of free capacities in a (gas) network

ξ	stochastic loads at exits
v	pressures & flows in network
u	free booking capacities at exits
$h = 0$	(P)DAE model describing (gas) transport in a network
$g \leq 0$	pressure bounds at exits
p	safety level
f	weighted sum of free capacities in the network

Examples

- How large are the pressure fluctuations in the network for common exit fluctuations and a particular switching scenario?
- How do the maximal booking capacities change with an additional pipeline/powerline?



Modeling of Gas Networks

(Simplified) parabolic model:

$$\partial_t \rho + \partial_x(\rho v) = 0,$$
$$\partial_x p = -\frac{\lambda}{2D}\rho v|v|$$

For the optimization:
On each pipe, we have the ODE

$$\partial_t p_R + \alpha \frac{q_R - q_L}{h} = 0,$$
$$\partial_t q_L + \beta \frac{p_R - p_L}{h} + \gamma \frac{q_L|q_L|}{p_R} = 0,$$

with constraints at junctions:

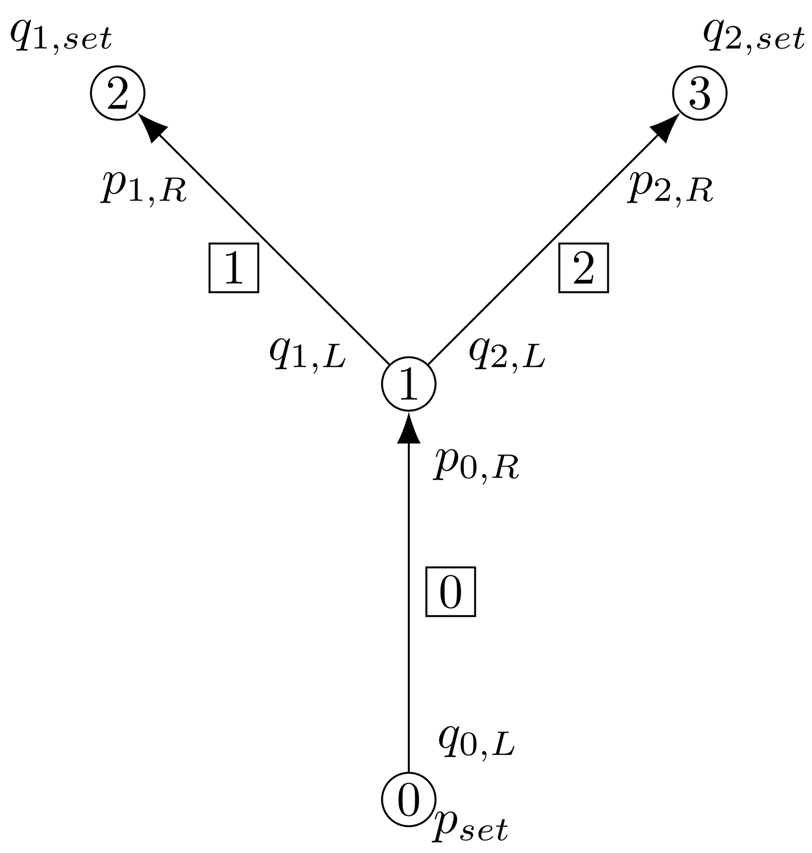
- sum of directed flows is zero
- pressures coincide

Time discretization via symplectic Euler:

$$p_{k+1} = p_k + \alpha\frac{\Delta t}{h} q_{k+1} - \alpha \frac{\Delta t}{h} q_{set}(t_{k+1}),$$
$$q_{k+1} = q_k - \beta\frac{\Delta t}{h} p_k + \beta\frac{\Delta t}{h} p_{set}(t_{k+1}) - \Delta t \gamma \frac{q_k|q_k|}{p_k},$$

with initial conditions

$$p_0 = \sqrt{p_{set}(t_0)^2 - \tilde{\gamma} q_{set}(t_0)|q_{set}(t_0)|},$$
$$q_0 = q_{set}(t_0)$$



Y-network for testing of spherical-radial decomposition

h	pipe length
Δt	time step
p_k, q_k	pressure/flow at time t_k
ρ	density
v	speed of the gas
λ, D	pipe parameters

Aims and Mathematical Challenges

Goal 1: Solution theory for SPDAEs in variational form

$$\mathcal{A}d(\mathcal{D}V_t) + \mathcal{B}(V_t)dt = C(t, V_t)dX_t$$

Methodology:

Combine (nonlinear) PDAE and SPDE analysis by

- decoupling and proper choice of X_t
- approximation of SPDAE solutions via SDAEs

Goal 2: Solve optimization problems with implicit probabilistic constraints

$$\min \left\{ f(u) \mid \mathcal{A}(\mathcal{D}v_\xi)' + \mathcal{B}(v_\xi) = r(\xi, \cdot),\ \mathbb{P}[g(u, v_\xi) \leq 0] \geq p \right\} \quad (p \in (0,1))$$

Methodology:

Apply spherical-radial decomposition to solve optimization problem

- with probabilistic constraints
- exploit perturbation theory for PDAEs

Past Results

Port-Hamiltonian systems of the form

$$\mathcal{A}^* \partial_t (\mathcal{A}u)(t) + \mathcal{B}u(t) = r(t)$$
$$u(0) = u_0$$

which arise from the Lagrange formulation of

$$a\partial_t p + \partial_x q = 0,$$
$$b\partial_t q + \partial_x p + dq = 0,$$

for $u = (p, q, \lambda) \in L^2(\mathcal{E}) \times H^1(\mathcal{E}) \times \mathbb{R}^{\nu_0}$ have an index of 2 for sufficiently smooth inputs

Plan for Next Year

1. Analyze (parabolic) prototype

$$\dot{x}(t) + f(z(t), X_t, t) = 0,$$
$$g(z(t), t) = 0,$$
$$dX_t + \mathcal{B}(X_t, t)dt + \mathcal{R}(X_t, z(t), t)dW_t = 0,$$
$$x(t_0) = x_0, \quad X_{t_0} = X_0$$

2. Apply spherical-radial decomposition to solve the optimization problem for the discretized system

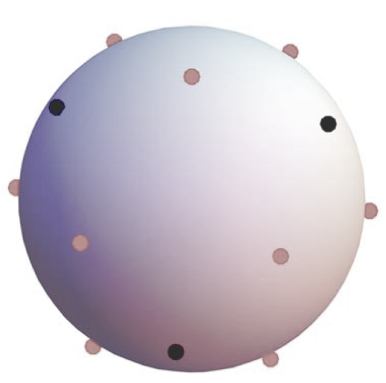
$$p_{k+1} = p_k + \alpha\frac{\Delta t}{h} q_{k+1} - \alpha \frac{\Delta t}{h} q_{set}(t_{k+1}),$$
$$q_{k+1} = q_k - \beta\frac{\Delta t}{h} p_k + \beta\frac{\Delta t}{h} p_{set}(t_{k+1}) - \Delta t \gamma \frac{q_k|q_k|}{p_k}$$

with $\xi \sim \mathcal{N}(0, LL^t)$ and therefore

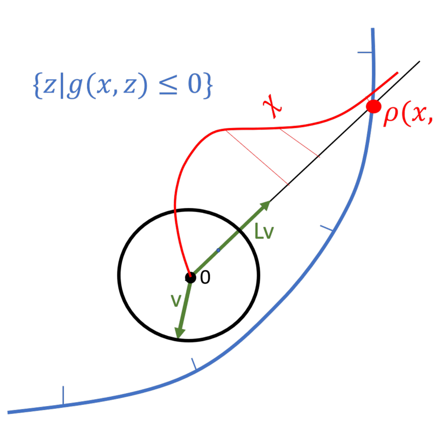
$$\mathbb{P}[g(x, \xi) \leq 0] = \int_{v \in \mathbb{S}^{m-1}} F_\chi(\rho(x, v)) d\mu_\zeta(v)$$

where

- $\mu_\zeta = \mathbb{P}_{\mathcal{U}(\mathbb{S}^{m-1})}$
- F_χ is the CDF of $\chi(m)$



QMC sampling on sphere



Spherical-radial decomposition

Cooperations

AA4-4

- coupling of supply/demand models with energy transport
- arbitrage constraints

AA4-5

- share approaches for modeling and optimization under uncertainty
- compare results

External

- DFG-TRR154 "Modellierung, Simulation und Optimierung am Beispiel von Gasnetzwerken"
- BMWi-Projektverbund MathEnergy "Mathematische Schlüsseltechniken für Energienetze im Wandel"

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