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Algebraische Gruppen / Liealgebren

Exercise Sheet 1

Exercise 1.1

Let $E = \mathbb{R}^2$, endowed with its standard scalar product. Let $m \geq 3$ be an integer and let $\mathcal{D}_m$ be the ('*dihedral*') group of all orthogonal endomorphisms of E which preserve a regular m-sided polygon centered at the origin.

(a) $\mathcal{D}_m$ has order $2m$. It consists of m reflections and m rotations. The rotations form a cyclic subgroup (of index 2).

(b) The group $\mathcal{D}_m$ is generated by reflections.

(c) The reflections form a single conjugacy class in $\mathcal{D}_m$ if m is odd, but form two classes if m is even.

(d) For which m does there exist a root system Φ in E such that $\mathcal{D}_m$ is the group generated by the reflections s_α for $\alpha \in \Phi$?

Exercise 1.2

Let $\Phi_1 \subset E_1$ and $\Phi_2 \subset E_2$ be root systems in the $\mathbb{R}$-vector spaces E_1, E_2. We put $E = E_1 \oplus E_2$ and identifty E_1 (resp. E_2) with the subspace $E_1 \oplus 0$ (resp. $0 \oplus E_2$) of E. In particular, we may regard Φ_1 and Φ_2 as subsets of E. Show that $\Phi_1 \cup \Phi_2$ is a root system in E. If Φ_1 and Φ_2 are reduced, then so is Φ.

Exercise 1.3

Let $\Phi \subset E$ be a root system in the euclidean $\mathbb{R}$-vector spaces E, with $W(\Phi)$-invariant scalar product $(.,.)$. Let $\alpha, \beta \in \Phi$.

(a) If $(\alpha, \beta) > 0$ and $\alpha \neq \beta$, then $\alpha - \beta \in \Phi$.

(b) If $(\alpha, \beta) < 0$ and $\alpha \neq -\beta$, then $\alpha + \beta \in \Phi$.

(c) If $\alpha - \beta \notin \Phi \cup \{0\}$ and $\alpha + \beta \notin \Phi \cup \{0\}$, then $(\alpha, \beta) = 0$.

Exercise 1.4

Let K be a field. A K-*algebra* is a K-vector space A together with a K-bilinear map (called multiplication)

$$A \times A \longrightarrow A, \quad (x, y) \mapsto xy.$$

It is called *associative* if $(xy)z = x(yz)$ for all $x, y, z \in A$, and if there is an element $1 \in A$ with $1x = x1 = x$ for all $x \in A$.

A K-algebra $\mathfrak{g}$ with multiplication map

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (x, y) \mapsto [x, y]$$

is called a Lie algebra over K if it satisfies the following conditions:

(i) For any $x \in \mathfrak{g}$ we have $[x, x] = 0$.

(ii) For any $x, y, z \in \mathfrak{g}$ we have

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{(Jacobi identity)}.$$

(a) A K-Lie algebra is never associative.

(b) Let $\mathfrak{g}$ be a K-Lie algebra. We have $[x, y] = -[x, y]$ for all $x, y \in \mathfrak{g}$.

(c) In the definition of a K-Lie algebra: If we replaced condition (i) by property (b), would that yield the same K-Lie algebras ?