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Algebraische Gruppen / Liealgebren

Exercise Sheet 2

Exercise 2.1

Let $\Phi \subset E$ be a root system, let $\alpha, \beta \in \Phi$ be linearly independent.

(a) The set $J = \{j \in \mathbb{Z} | \beta + j\alpha \in \Phi\}$ is an interval of the form $[-q, p] \cap \mathbb{Z}$ for certain $p, q \in \mathbb{Z}_{\geq 0}$. (Hint: Exercise 1.3)

(b) The set of elements of Φ of the form $\beta + j\alpha$ with $j \in \mathbb{Z}$ is stable under the reflection s_α. We have $s_\alpha(\beta + p\alpha) = \beta - q\alpha$.

(c) $q - p = \check{\alpha}(\beta)$.

Exercise 2.2

Let $\Phi \subset E$ be a root system, let $\alpha, \beta \in \Phi$, let $s_{\check{\alpha}}, s_{\check{\beta}} \in W(\Phi)$ be the corresponding reflections. In the lectures we saw the explicit list of all the possibilities for the relative position of α, β. In each of these cases find the order of the composition $s_\alpha s_\beta = s_\alpha \circ s_\beta \in W(\Phi)$.

If $\check{\alpha}(\beta) = 2$ and $\check{\beta}(\alpha) = 1$, and similarly if $\check{\alpha}(\beta) = -2$ and $\check{\beta}(\alpha) = -1$, we get $\mathrm{ord}(s_\alpha s_\beta) = 4$ (again by evaluating on α and β).

If $\check{\alpha}(\beta) = 3$ and $\check{\beta}(\alpha) = 1$, and similarly if $\check{\alpha}(\beta) = -3$ and $\check{\beta}(\alpha) = -1$, we get $\mathrm{ord}(s_\alpha s_\beta) = 6$ (again by evaluating on α and β).

If α and β are proportional then $\mathrm{ord}(s_\alpha s_\beta) = 1$.

Exercise 2.3

(a) Show that the list of one- and two-dimensional root systems given in paragraph two of the lectures indeed consists of root systems, and is complete.

(b) Find the orders of the corresponding Weyl groups.

Exercise 2.4

Let $\Phi \subset E$ be a root system. Show that $W(\Phi)$ is a normal subgroup in $A(\Phi)$.

Exercise 2.5

Let $\Phi \subset E$ be a reduced root system.

(a) Let $E' \subset E$ be an $\mathbb{R}$-sub vector space, let $\alpha \in \Phi$ be such that $s_\alpha(E') \subset E'$. Show that $\alpha \in E'$ or $E' \subset \{v \in E \mid s_\alpha(v) = v\}$.

(b) Show that $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$ is a reduced root system in E^* and that there is a canonical isomorphism $W(\Phi) \cong W(\check{\Phi})$.