Prof. Dr. Elmar Große-Klönne
Lehrstuhl für Algebraische Zahlentheorie

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Algebraische Gruppen / Liealgebren

Exercise Sheet 3

Exercise 3.1

In the situation of Exercise 2.1 we call the set $\{\beta + j\alpha | j \in J\}$ the α-string of roots passing through β. The number $p + q$ (i.e. the cardinality of this string minus 1) is called the length of this string.

(a) Show that this length is 0, 1, 2 or 3.

(b) Show (by giving examples) that each of these values (0, 1, 2 or 3) can occur.

Exercise 3.2

Let $n \geq 1$, let $e_1, \ldots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$. Let

$$E = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 0\},$$

$$\Phi = \{e_i - e_j \mid 1 \leq i \neq j \leq n + 1\}$$

(thus Φ consists of $n(n + 1)$ elements of E). Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots, \quad \alpha_n = e_n - e_{n+1}.$$

(a) $\Phi \subset E$ is a root system. It is denoted by A_n.

(b) Δ is a basis for Φ.

(c) $W(\Phi)$ is isomorphic with the symmetric group S_{n+1} on $n + 1$ elements.

Exercise 3.3

Let K be a field. For a K-Lie algebra $\mathfrak{g}$ (with Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, cf. Exercise 1.4) and an element $x \in \mathfrak{g}$ we define the K-linear endomorphism

$$\mathrm{ad}(x) : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad y \mapsto \mathrm{ad}(x)(y) := [x, y].$$

Now consider

$$\mathfrak{sl}_2(K) = \{A \in \mathrm{Mat}_{2,2}(K) \mid \mathrm{tr}(A) = 0\}$$

(with $\mathrm{tr}(A)$ denoting the trace of A). Put

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $\mathfrak{sl}_2(K)$ is a K-Lie algebra with respect to the Lie bracket

$$[A, B] := AB - BA.$$

(b) Show that X, Y, H is a K-basis for (the K-vector space underlying) $\mathfrak{sl}_2(K)$.

(c) Find the matrices of the K-linear endomorphisms $\mathrm{ad}(X)$, $\mathrm{ad}(Y)$ and $\mathrm{ad}(H)$ with respect to this K-basis. Find the eigenvalues of $\mathrm{ad}(H)$.