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Algebraische Gruppen / Liealgebren

Exercise Sheet 4

Exercise 4.1

Let $\Phi \subset E$ be an irreducible reduced root system, let $(.,.)$ be a $W(\Phi)$-invariant scalar product on E. Let $\Delta \subset \Phi$ be a basis of Φ.

Recall the length $\ell(w)$ defined for $w \in W(\Phi)$. Theorem 15 in par. 4 says (and you may use this):

$$\ell(w) = |\{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}|.$$

(a) Show that $\check{\Delta} = \{\check{\alpha} \in \check{\Phi} \mid \alpha \in \Delta\}$ is a basis of the dual root system $\check{\Phi} \subset E^*$ (cf. Exercise 2.5).

(b) Let $\ell : W(\Phi) \to \mathbb{Z}_{\geq 0}$ denote the length function (with respect to Δ). Let $w \in W(\Phi)$, written (not necessarily reduced) as $w = s_{\alpha_1} \cdots s_{\alpha_r}$ with $\alpha_i \in \Delta$ (not necessarily pairwise distinct). Show that $(-1)^r = (-1)^{\ell(w)}$.

(c) Show that there exists exactly one element $w_0 \in W(\Phi)$ with $w_0(\Phi^+) = \Phi^-$.

(d) What can you say about $\ell(w_0)$? Show that $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for all $w \in W(\Phi)$.

(e) Let $w_0 = s_{\alpha_1} \cdots s_{\alpha_r}$ (with $\alpha_i \in \Delta$) be a reduced expression of the element $w_0 \in W(\Phi)$ considered in (c). Show that each $\alpha \in \Delta$ occurs among the α_i at least once.

Exercise 4.2

Let $\Phi \subset E$ be a root system, let $\alpha, \beta \in \Phi$ be linearly independent and assume that also $\alpha + \beta \in \Phi$. Define $p, q \in \mathbb{Z}$ as in Exercise 2.1. Show that

$$\frac{(\beta + \alpha, \beta + \alpha)}{(\beta, \beta)} = \frac{q+1}{p}.$$

Hint: A case by case study.

Exercise 4.3

Let $n \geq 2$, let $e_1, \ldots, e_n$ be the standard basis of $E = \mathbb{R}^n$. Let

$$\Phi = \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$$

(thus Φ consists of $2n + 2n(n-1) = 2n^2$ elements). Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_n.$$

(a) $\Phi \subset E$ is a root system. It is denoted by B_n.

(b) Δ is a basis for Φ.

(c) $W(\Phi)$ is isomorphic with the semidirect product of the symmetric group S_n and $(\mathbb{Z}/2\mathbb{Z})^n$, with $(\mathbb{Z}/2\mathbb{Z})^n$ normal in $W(\Phi)$.