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Algebraische Gruppen / Liealgebren

Exercise Sheet 5

Exercise 5.1

Let $n \geq 2$, let $e_1, \ldots, e_n$ be the standard basis of $E = \mathbb{R}^n$. Let

$$\Phi = \{\pm 2e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$$

(thus Φ consists of $2n + 2n(n-1) = 2n^2$ elements). Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n.$$

(a) $\Phi \subset E$ is a root system. It is denoted by C_n.

(b) Δ is a basis for Φ.

(c) $W(B_n) \cong W(C_n)$.

(d) $C_n \cong B_n$ if and only if $n = 2$.

Exercise 5.2

Let V be a finite dimensional real vector space. Let Φ be a finite set and suppose that for all $\alpha \in \Phi$ we are given a hyperplane H_α in V. For $v \in V - H_\alpha$ we denote by $D_\alpha(v)$ the connected component of $V - H_\alpha$ which contains v. Consider the equivalence relation $\equiv$ on V for which $v \equiv v'$ if and only if for all $\alpha \in \Phi$ we have: $v, v' \in H_\alpha$ or $[v, v' \notin H_\alpha$ and $D_\alpha(v) = D_\alpha(v')]$. The equivalence classes for this equivalence relation are called the *sides* of the hyperplane arrangement $\mathcal{H} = \{H_\alpha \mid \alpha \in \Phi\}$. For a side F we call

$$L_F = \bigcap_{H_\alpha \supset F} H_\alpha$$

the *support* of F (if F is not contained in any H_α we put $L_F = V$), and we call $\dim(L_F)$ the dimension of F.

Let F be a side.

(a) Let $x \in F$. We have $F = L_F \cap \bigcap_{\substack{\alpha \in \Phi \\ x \notin H_\alpha}} D_\alpha(x)$ and $\overline{F} = L_F \cap \bigcap_{\substack{\alpha \in \Phi \\ x \notin H_\alpha}} \overline{D_\alpha(x)}$ (where $\overline{(.)}$ denotes the operation of taking the topological closure in the topological vector space V).

(b) F is a convex open subset of L_F.

(c) The closure $\overline{F}$ of F in V is the union of F and some sides of strictly smaller dimension.

(d) Let also F' denote a side. If $\overline{F} = \overline{F}'$ then $F = F'$.

(e) Let $\Psi \subset \Phi$ be a subset, put $M = \cap_{\alpha \in \Psi} H_\alpha$. Then the following (i), (ii), (iii) are equivalent: (i) There is a side F' with $M = L_{F'}$ and $F' \cap \overline{F} \neq \emptyset$. (ii) There is a side F' with $M = L_{F'}$ and $F' \subset \overline{F}$. (iii) There is some $x \in M \cap \overline{F}$ with the following property: for all $\alpha \in \Phi$ with $x \in H_\alpha$ we have $M \subset H_\alpha$.

Exercise 5.3

Let Φ be a reduced root system, let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for Φ. For $1 \leq i, j \leq n$ let $m_{ij} = \mathrm{ord}(s_{\alpha_i} s_{\alpha_j})$ be the order of the element $s_{\alpha_i} s_{\alpha_j} = s_{\alpha_i} \circ s_{\alpha_j}$ in $W(\Phi)$. We define the *Coxeter matrix* of Φ to be the $n \times n$-matrix $M = (m_{ij})_{1 \leq i,j \leq n}$.

(a) M is symmetric, all diagonal entries are $= 1$, we have $m_{ij} \in \{2, 3, 4, 6\}$ for $i \neq j$.

(b) Up to reindexing, M does not depend on the choice of Δ.

(c) One can read off from M the number of the irreducible direct summands of Φ, and their ranks.

(d) Φ is uniquely determined by M.