Algebraische Gruppen / Liealgebren

Exercise Sheet 6

Exercise 6.1

Let $n \geq 4$, let $e_1, \ldots, e_n$ be the standard basis of $E = \mathbb{R}^n$. Let $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$ (thus Φ consists of $2n(n-1)$ elements). Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, ..., $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$.

(a) $\Phi \subset E$ is a root system. It is denoted by D_n.

(b) Δ is a basis for Φ.

(c) $W(\Phi)$ is isomorphic with the semidirect product of the symmetric group S_n and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$.

Exercise 6.2

Let $\mathcal{H} = \{H_\alpha \mid \alpha \in \Phi\}$ be a hyperplane arrangement in the real vector space V, as in Exercise 5.2. A connected component of $V - \cup_{\alpha\in\Phi} H_\alpha$ is called a *chamber* for $\mathcal{H}$. Let L be a hyperplane in V, let $\Omega \subset L$ be a non empty subset, open in L.

(a) The sides which are not contained in any H_α are precisely the chambers. Any element of V lies in the closure of some chamber.

(b) If $L \neq H_\alpha$ for all $\alpha \in \Phi$ then there is a chamber C with $C \cap \Omega \neq \emptyset$.

(c) If $L = H_\alpha$ for some $\alpha \in \Phi$ then there is some $x \in \Omega$ not contained in any other H_β (i.e. with $x \notin H_\beta$ for all $\beta \in \Phi$ with $H_\beta \neq L$).

Exercise 6.3

Let Φ be a reduced irreducible root system. Recall that also $\check{\Phi}$ is a recuded root system (cf. Exercise 2.5).

(a) $\check{\Phi}$ is irreducible.

(b) If all $\alpha \in \Phi$ have the same length, then also all $\check{\alpha} \in \check{\Phi}$ have the same length, and Φ is isomorphic (as a root system) with $\check{\Phi}$.

(c) If Φ contains elements of two different lengths, short roots and long roots, then this is also true for $\check{\Phi}$, and $\alpha \in \Phi$ is long if and only if $\check{\alpha} \in \check{\Phi}$ is short.

Exercise 6.4

Let K be a field. Describe all isomorphism classes of K-Lie algebras $\mathfrak{g}$ with $\dim_K(\mathfrak{g}) \leq 2$.