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Zahlenentheorie

Exercise Sheet 1

Exercise 1.1

(a) An absolute value on a finite field is trivial.

(b) An absolute value $|.|$ on a field is non archimedean if and only if there is some $M > 0$ with $|n \cdot 1| \leq M$ for all $n \in \mathbb{Z}$.

(c) An absolute value on a field of positive characteristic is non archimedean.

Exercise 1.2

(Ostrowski's Theorem)

Let $|.|$ be a non trivial absolute value on the field $\mathbb{Q}$ of rational numbers. Show that one of the following two alternatives must hold true:

(a) $|.|$ is a p-adic absolute value, i.e. there is a prime number p and some $c \in \mathbb{R}$ with $0 < c < 1$ such that $|x| = c^{v_p(x)}$ for all $x \in \mathbb{Q}$. (Here $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is the p-adic valuation.)

(b) There is some $\alpha > 0$, such that $|x| = |x|_\infty^\alpha$ for all $x \in \mathbb{Q}$, where $|.|_\infty : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is the usual (archimedean) absolute value.

Ansatz: If $|.|$ is non archimedean, then $\{x \in \mathbb{Z} \,;\; |x| < 1\}$ is a non trivial prime ideal in $\mathbb{Z}$. More difficult is the case where $|.|$ is archimedean. First show $|m| \leq \max\{1, |n|\}^{\log(m)/\log(n)}$ for some $n, m \in \mathbb{Z}$ with $n, m > 1$. [To do this: For $t \in \mathbb{N}$ write $m^t = \sum_{i=0}^{s} a_i n^i$ with $a_i \in \{0, 1, \ldots, n-1\}$ and $a_s \neq 0$; we then have $s/t \leq \log(m)/\log(n)$. On the other hand, the triangle inequality implies $|m|^t \leq (s+1)n \cdot \max\{1, |n|^s\}$.] Since $|.|$ is archimedean we must have $|n| > 1$ (cf. Problem 1). Now swap the roles of m and n in the resulting inequality.

Exercise 1.3

(a) Let L/K be an algebraic field extension, let $|.|$ be an absolute value on L. Show that $|.|$ is trivial if and only if the restriction of $|.|$ to K is trivial.

(b) There exists one and only one absolute value on the field of complex numbers, which extends the usual archimedean absolute value on the field of real numbers.

Exercise 1.4

(Gauss's Lemma)

Let v be a discrete valuation on a field K, put $A = \{x \in K \mid v(x) \geq 0\}$. Let $f, g \in K[X]$ be normed (i.e. monic, i.e. with leading coefficient $= 1$) polynomials. Show that if $fg \in A[X]$ then also $f \in A[X]$ and $g \in A[X]$. How does this imply Gauss's Lemma in its 'usual' form (with $A = \mathbb{Z}$ and $K = \mathbb{Q}$) ?