The power operation in the Galois cohomology of a reductive group over a number field

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Abstract: For a number field K admitting an embedding into the field of real numbers $\mathbb{R}$, it is impossible to construct a functorial in G group structure in the Galois cohomology pointed set $H^1(K,G)$ for all connected reductive K-groups G. However, over an arbitrary number field K, we define a *diamond* (or *power*) operation of raising to power n

$$(x,n) \mapsto x^{\Diamond n} : H^1(K,G) \times \mathbb{Z} \longrightarrow H^1(K,G).$$

We show that this operation has many functorial properties. When G is a torus, the set $H^1(K,G)$ has a natural group structure, and $x^{\Diamond n}$ coincides with the n-th power of x in this group.

For a cohomology class x in $H^1(K,G)$, we define the period $\mathrm{per}(x)$ to be the least $n > 0$ such that $x^{\Diamond n} = 1$, and the index $\mathrm{ind}(x)$ to be the greatest common divisor of the degrees $[L:K]$ of finite separable extensions L/K splitting x. These period and index generalize the period and index of a central simple algebra over K (in the special case where G is the projective linear group PGL_n, the elements of $H^1(K,G)$ can be represented by central simple algebras). For an arbitrary reductive group G defined over a local or global field K, we show that $\mathrm{per}(x)$ divides $\mathrm{ind}(x)$, that $\mathrm{per}(x)$ and $\mathrm{ind}(x)$ have the same prime factors, but the equality $\mathrm{per}(x) = \mathrm{ind}(x)$ may not hold.

The talk is based on a joint work with Zinovy Reichstein. All necessary definitions will be given, including the definition of the Galois cohomology set $H^1(K,G)$.