S^6 AND THE GEOMETRY OF NEARLY KÄHLER 6-MANIFOLDS

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ABSTRACT. We review results on and around the almost complex structure on S^6, both from a classical and a modern point of view. These notes have been prepared for the Workshop "(Non)-existence of complex structures on S^6" (*Erste Marburger Arbeitsgemeinschaft Mathematik – MAM-1*), held in Marburg in March 2017.

1. INTRODUCTION

It is well known that the sphere S^6 admits an almost Hermitian structure induced by octonionic multiplication, and that this structure stems from the transitive action of the compact exceptional Lie group G_2 on it. In 1955, Fukami and Ishihara were presumably the first authors to devote a separate paper to the detailed investigation of S^6 and showed in particular that S^6 is the naturally reductive space $G_2/SU(3)$ [FI55]. In 1958, Calabi studied hypersurfaces in the space of imaginary octonions and proved that the induced almost complex structure is never integrable if the hypersurface is compact [Cal58]. In fact the almost Hermitian structure on the 6-sphere is a very special one: Already in [FI55], it is observed that the Levi Civita derivative of J satisfies

$$(*) \qquad\qquad (\nabla^g_X J)X \;=\; 0 \text{ for all vector fields } X.$$

Such manifolds are called *nearly Kähler* and they were investigated intensively by Gray in a series of papers [Gr66, Gra70, G76]. In particular, he showed in dimension 6 that they are Einstein and their first Chern class vanishes. In fact, for many reasons dimension 6 is of particular interest for nearly Kähler geometry [Na10]. For a long time the only compact examples of nearly Kähler manifolds were the four homogeneous examples: S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold F_2. The aim of this paper is to provide a concise review of properties of nearly Kähler manifolds in dimension 6 with special attention given to the sphere S^6. After some historical remarks, we start by recalling Calabi's result about hypersurfaces in the space of imaginary octonions $\mathbb{R}^7$. Then we discuss the intrinsic torsion approach and naturally reductive spaces and briefly recall Gray's [G76] and Kirichenko's [Ki77] results. Next we present the spinoral approach of R. Grunewald [Gru90] and a modern view on it. We finish by giving an overview of L. Foscolo and M. Haskins contribution [FH17]. They discovered non-homogeneous cohomogeneity one nearly Kähler structures on S^6 and conjectured that these are the only cohomogeneity one examples.

2. SOME HISTORICAL COMMENTS

Clearing the facts around the almost complex structure on S^6 took several independent steps. In particular, it was not noticed immediately that (and how) it was related to the transitive action of G_2.

Montgomery and Samelson proved in 1943 that the only compact connected simple Lie group which can be transitive on S^{2n} is $SO(2n + 1)$—*except for a a finite number of $n's$* [MS43, Thm II, p.462]. Their method was of topological nature and required the knowledge of the homology rings of simple Lie groups, which was not yet available for the five exceptional simple Lie groups; hence they couldn't give any further information on the exceptional cases.

Six years later, Armand Borel proceeded by constructing the homogeneous spaces directly, which lead him to the result that the only sphere with a transitive group G acting that is not orthogonal is S^6 with $G = G_2$ [Bo49, Thm III, p. 586]. This completed the classification of transitive sphere actions and showed, in particular, that G_2 is the only exceptional Lie group with such an action.

Meanwhile, Adrian Kirchhoff had noticed in 1947 that S^6 carries an almost complex structure induced from octonionic multiplication ([Ki47]; see also [Eh50]). In his main theorem, Kirchhoff related the existence of an almost complex structure on S^n to the parallelism of S^{n+1}.

In 1951, Ehresmann and Libermann [EL51] as well as Eckmann and Frölicher [EF51] observed independently that this almost complex structure on S^6 is not integrable — in fact, their articles appeared in the same volume of the Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris. While Eckmann and Frölicher were interested in formulating the integrability condition and treated S^6 merely as an example where it didn't hold, the aim of Ehresmann and Libermann was the local description of locally homogeneous almost hermitian manifolds in terms of Cartan structural equations, and they found that the equations exhibited an exceptional structure for $n = 6$. They stated [EL51, p. 1282]:

> "La structure considérée est donc localement équivalente à une structure presque hermitienne sur S^6 admettant G_2 comme groupe d'automorphismes. Ce groupe ne peut laisser invariante sur S^6 qu'une seule structure presque complexe. Celle-ci est donc isomorphe à la structure presque complexe définie à l'aide des octaves de Cayley. Comme la deuxième torsion dans les formules (5) n'est pas nulle, cette structure ne dérive pas d'une structure complexe."[1]

Hence, they seem to be the first authors to connect the transitive G_2-action on S^6 to its octonionic almost complex structure. A detailed account of the results of [EF51] and further material was given by Frölicher four years later [Fr55]; however, in his discussion of homogeneous almost complex manifolds, he doesn't mention S^6. Remarkably, he described already (as did [FI55]) the characteristic connection of S^6 and proved that its torsion is given by the Nijenhuis tensor.

In [FI55], all these thoughts on S^6 are brought together for the first time, and the characteristic connection is proved to coincide with the canonical connection of the homogeneous space $G_2/SU(3)$.

The first author to suggest the investigation of manifolds satisfying the abstract nearly Kähler condition ($*$) was Tachibana in [Ta59], who called such manifolds K-spaces and proved, amongst other things, that their Nijenhuis tensor is totally antisymmetric. No

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[1]"The structure we considered is therefore locally equivalent to an almost hermitian structure on S^6 admitting G_2 as its group of automorphisms. This group can leave invariant only one almost complex structure on S^6. It is therefore isomorphic to the almost complex structure defined with the help of Cayley's octonions. Since the second torsion of the formulas (5) doesn't vanish, this structure is not induced from a complex structure." (translated by the authors)

examples were discussed, although it is clear from the reference made to [FI55, Fr55] that the inspiration came from S^6. The paper [Ko60] continued the investigation of K-spaces.

Inspired by the papers of Calabi [Cal58] and Koto [Ko60], Alfred Gray used in 1966 for the first time the term *nearly Kähler manifold* [Gr66]. He writes in the introduction:

> "The manifolds we discuss include complex and almost Kähler manifolds; also S^6 with the almost complex structure derived from the Cayley numbers falls into a class of manifolds which we call nearly Kählerian."

This was the starting point of the career of S^6 as a most remarkable nearly Kähler manifold. Surprisingly, most classes of almost Hermitian manifolds that were systemized later in the Gray-Hervella classification [GH80] appear already in this paper.

3. The almost complex structures induced from octonions

In this section we present an explicit approach for constructing the nearly Kähler structure on S^6. The construction goes back to Calabi [Cal58], who studied hypersurfaces in $\mathbb{R}^7$ with a complex structure induced from octonions.

3.1. Seven-dimensional cross products.
Recall that the octonion algebra $\mathbb{O}$ is the unique 8-dimensional composition algebra (or equivalently normed division algebra). It can be defined from quaternions using the Cayley-Dickson construction $\mathbb{O} = \mathbb{H} \oplus J\mathbb{H}$, with the following operations:

$$\overline{q_1 + Jq_2} = \overline{q_1} - Jq_2, \quad (q_1 + Jq_2)(q_3 + Jq_4) = q_1 q_3 + \overline{q_4} q_2 + J(q_2 \overline{q_3} + q_4 q_1).$$

Consequently, the octonions can be viewed as an 8 dimensional (non-associative) algebra with basis $1, e_1, \ldots, e_7$, where e_i are imaginary units and the multiplication between them is defined above. Note that we can take e_1, e_2, e_3 to be imaginary quaternions. Consider the vector subspace of imaginary octonions $\mathcal{Y} := span\{e_1, \ldots, e_7\}$ which, as a vector space, is isomorphic to $\mathbb{R}^7$. The octonion multiplication induces (by restriction and projection) a cross product on $\mathcal{Y}$ via the formula

$$A \times B := \frac{1}{2}(AB - BA).$$

The group of automorphisms of $\mathcal{Y}$ is the exceptional group G_2, see [Dr17]. The vector space $\mathcal{Y}$ together with a cross product and scalar product is sometimes called the Cayley space. The cross product has the following properties, see again [Dr17] :

- $\langle A, (B \times C) \rangle = \langle (A \times B), C \rangle =: (ABC)$ (*scalar triple product identity*),
- $A \times (A \times B) = -|A|^2 B + \langle A, B \rangle A$ (*Lagrange* or *Malcev identity*),
- If C is orthogonal to A and B, then

$$(A \times B) \times C = A \times (B \times C) - \langle A, B \rangle C.$$

However, there are some significant differences between dimensions 3 and 7 coming from the non-associativity of octonions—for example, the Jacobi identity does not hold in dimension 7.

3.2. **Almost complex structure on hypersurfaces in Cayley space.** The results described in this section are mainly due to Calabi [Cal58]. Let S be a 6-dimensional oriented manifold immersed into the Cayley space $\mathcal{Y}$. The canonical orientation on $\mathcal{Y}$ induces a normal vector field on S, called N. Consider its second fundamental form and denote by K its shape operator. The eigenvalues of K are just the principal curvatures. We define $J \in End(TS)$ by

$$J(X) := N \times X, \quad X \in TS,$$

and take g to be the metric on S induced by the pull back of the scalar product on $\mathcal{Y}$.

Lemma 3.1. *J is an almost complex structure on S such that (S, J, g) is an almost Hermitian manifold, and it satisfies the identity*

$$\langle (\nabla_X^g J)(Y), Z \rangle = \langle K(X) \times Y, Z \rangle.$$

Proof. First we need to show that $J^2 = -\,id$. By the Malcev identity, $N \times (N \times X) = -|N|^2 X + \langle N, X \rangle N$, for any $X \in (TS)$. As X is perpendicular to N, this proves the claim. Then, using the scalar triple product and Malcev identities we obtain

$$\langle J(X), J(Y) \rangle = \langle N \times X, N \times Y \rangle = \langle (N \times X) \times N, Y \rangle = -\langle N \times (N \times X), Y \rangle = \langle X, Y \rangle \,,$$

showing the J-invariance of the metric. Finally, let us compute $(\nabla_X^g J)(Y)$ for any tangent vectors X, Y:

$$
\begin{aligned}
(\nabla_X^g J)(Y) &= \nabla_X^g (J(Y)) - J(\nabla_X^g Y) = \nabla_X^g (N \times Y) - N \times \nabla_X^g Y \\
&= \nabla_X^g N \times Y + N \times \nabla_X^g Y - N \times \nabla_X^g Y = K(X) \times Y \,.
\end{aligned}
$$

This yields the claimed formula for $\nabla_X^g J$. $\qquad\qquad\square$

Definition 3.1. Let (M, g, J) be an almost Hermitian manifold. If $(\nabla_X^g J)X = 0$ and $\nabla^g J \neq 0$, then (M, g, J) is called nearly Kähler manifold.

The almost complex structure of a nearly Kähler manifold is never integrable, see [G76] or [AFS05]. In fact, an easy calculation shows that its Nijenhuis tensor is given by $N(X, Y) = 4(\nabla_X^g J)JY$. A direct consequence of the last lemma is the next

Proposition 3.1. *Let g be the standard metric on S^6 and J the almost complex structure defined by the cross product on the Cayley space. Then (S^6, g, J) is a nearly Kähler manifold.*

Proposition 3.2 (Calabi [Cal58]). *If (S, I) is a compact, oriented 6-manifold with an immersion into the Cayley space $\mathcal{Y}$, the induced almost complex structure J is non-integrable.*

To prove this theorem Calabi studied the shape operator K of the hypersurface. He found that the integrability condition for J is that K is complex anti-linear,

$$K \circ J = -J \circ K \,.$$

In any closed hypersurface of the Euclidian space, there exists an open subset on which the second fundamental form is positive or negative definite. But if the tangent vector X is an eigenvector of the shape operator with eigenvalue λ, then $J(X)$ is again an eigenvector with eigenvalue $-\lambda$. This yields the contradiction.

4. S^6 AS NATURALLY REDUCTIVE SPACE

A homogeneous Riemannian space $M = G/H$ is called reductive if there exists an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Denote by $\langle .., .. \rangle$ the inner product in $\mathfrak{m}$ defining the G-invariant metric. If it satisfies

$$\langle [X, Y], Z \rangle \;=\; - \langle [X, Z], Y \rangle \,,$$

the homogeneous space is called naturally reductive. In this case the tensor

$$T(X, Y, Z) \;:=\; \langle [X, Y], Z \rangle$$

is totally skew symmetric, i.e. T is a 3-form. The canonical connection ∇^c of G/H is the unique metric connection with skew symmetric torsion tensor T,

$$\nabla^c \;=\; \nabla^g \;+\; \frac{1}{2} T \,.$$

The holonomy of ∇^c is contained in the isotropy group H, i.e. the canonical connection is much more adapted to the space G/H then the Levi-Civita connection ∇^g. A naturally reductive space with vanishing torsion T is a Riemannian symmetric space. Naturally reductive homogeneous spaces have the special property that the torsion and the curvature are parallel with respect to the canonical connection,

$$\nabla^c \mathcal{R}^c \;=\; 0 \,, \quad \nabla^c T^c \;=\; 0 \,.$$

Observe that a homogenous Riemannian manifold can be naturally reductive in different ways, it depends on the choice of the subgroup $G \subset Iso(M)$ of the isometry group. However, this happens only for spheres or Lie groups, see the recent results of C. Olmos and S. Reggiani [OR12].

Any nearly Kähler manifold admits a unique hermitian connection with skew symmetric torsion, too. This connection has been introduced by A. Gray and is called the characteristic connection of the nearly Kähler manifold, see [G76]. Let us explain the proof as well as the formula for the characteristic torsion.

Proposition 4.1. *Let (M, g, J) be a nearly Kähler manifold. There exists a unique metric connection preserving the almost complex structure and with skew symmetric torsion, and its torsion 3-form is given by the formula*

$$T^c(X, Y, Z) \;=\; \langle (\nabla^g_X J)(JY), Z \rangle \,.$$

Proof. Consider a metric connection $\nabla = \nabla^g + \frac{1}{2} T$ with skew symmetric torsion. The condition $\nabla J = $ yields the equation

$$0 \;=\; \langle (\nabla^g_X J)(Y), Z \rangle + \frac{1}{2} T(X, JY, Z) + \frac{1}{2} T(X, Y, JZ) \,.$$

Symmetrizing the latter equation with respect to X and Y, we obtain

$$0 \;=\; \langle (\nabla^g_X J)(Y), Z \rangle + \langle (\nabla^g_Y J)(X), Z \rangle + \frac{1}{2} T(X, JY, Z) + \frac{1}{2} T(Y, JX, Z) \,.$$

In case the almost complex structure is nearly Kähler we obtain the condition

$$T(X, JY, Z) \;=\; -T(Y, JX, Z),$$

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and moreover

$$T(X, Y, JZ) \;=\; T(Y, JZ, X) \;=\; -T(Z, JY, X) \;=\; T(X, JY, Z) \,.$$

Inserting the latter formula into the first one, we finally obtain the formula for the characteristic torsion

$$0 \;=\; \langle (\nabla^g_X J)(Y)\,,\, Z \rangle \;+\; T(X, JY, Z) \,.$$

$\square$

If a nearly Kähler manifold is a reductive homogeneous space, the canonical connection in the sense of reductive spaces coincides with the characteristic connection in the sense of nearly Kähler manifolds. With respect to the full isometry group $G = \mathrm{SO}(7)$, the round sphere S^6 becomes a symmetric space,

$$S^6 \;=\; \mathrm{SO}(7)/\mathrm{SO}(6),$$

the canonical connection coincides with the Levi-Civita connection and is hence torsion free. On the other side, taking into account the almost complex structure J induced by the octonions, S^6 becomes a naturally reductive space (see [FI55], [R93]). Indeed, by construction J and g are invariant under the action of the automorphism group of octonions. In particular, the exceptional group $G_2 \subset SO(7)$ preserves the metric as well as the almost complex structure. Moreover, G_2 acts transitively on S^6. The isotropy subgroup preserves the linear complex structure of the tangent space. Consequently, its an 8-dimensional subgroup of U(3) isomorphic to SU(3), see [Dr17].

Proposition 4.2. $S^6 = G_2/\mathrm{SU}(3)$ *is naturally reductive and the torsion* 3*-form of the canonical connection is given by the formula*

$$T^c(X, Y, Z) \;=\; -\langle J(X \times Y), Z \rangle \;=\; -\langle N, (X \times Y) \times Z \rangle \,.$$

Proof. Consider the metric connection ∇ defined by the formula

$$\langle \nabla_X Y, Z \rangle \;:=\; \langle \nabla^g_X Y, Z \rangle \;-\; \frac{1}{2} \langle J(X \times Y), Z \rangle \,.$$

We compute the covariant derivative ∇J :

$$\langle (\nabla_X J)Y, Z \rangle \;=\; \langle (\nabla^g_X J)Y, Z \rangle \;-\; \frac{1}{2} \langle J(X \times Y), Z \rangle \;+\; \frac{1}{2} \langle J \circ J(X \times Y), Z \rangle \,.$$

Next we apply the formula for the covariant derivative $\nabla^g J$. Then we obtain

$$2 \, \langle (\nabla_X J)Y, Z \rangle \;=\; \langle X \times Y, Z \rangle \;-\; \langle J(X \times JY), Z \rangle \,.$$

Since N and X, Y are orthogonal, the sume of the right side vanishes. Indeed, we have

$$\langle J(X \times JY), Z) \rangle \;=\; \langle N \times (X \times (N \times Y)), Z \rangle \;=\; -\langle N \times (N \times (X \times Y)), Z \rangle \;=\; \langle (X \times Y, Z \rangle \,.$$

The computation proves that the connection ∇ is a metric connection preserving the almost complex structure J. Moreover, its tosion

$$T^c(X, Y, Z) \;=\; -\langle J(X \times Y), Z \rangle \;=\; -\langle N, (X \times Y) \times Z \rangle$$

is skew symmetric. Consequently, ∇ is the canonical (characteristic) connection of the naturally reductive and nearly Kähler space $G_2/\mathrm{SU}(3)$. $\square$

By a theorem of Butruille [B05], no other nearly Kähler structure on S^6 can be homogeneous. In fact he proved that the only homogeneous nearly Kähler 6-manifolds are S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold F_2 with their standard metrics.

5. G-STRUCTURES AND CONNECTIONS WITH SKEW SYMMETRIC TORSION

The connection defined above can be described from the point of view of G-structures. Let G be a closed Lie subgroup of $SO(n)$ and let $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ be the corresponding orthogonal decomposition of the Lie algebra $\mathfrak{so}(n)$. Then a G-structure on Riemannian manifold M is a reduction $\mathcal{R}$ of its frame bundle, which is principal $SO(n)$-bundle, to the subgroup G. As the Levi-Civita connection is a 1-form with values in $\mathfrak{so}(n)$, using the decomposition $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, we get a direct sum decomposition of its restriction to $\mathcal{R}$ into a connection in principal G-bundle $\mathcal{R}$ and a term Γ corresponding to $\mathfrak{m}$. Γ is a 1-form on M with values in the associated bundle $\mathcal{R} \times_G \mathfrak{m}$ and is called the intrinsic torsion. It measures the integrability of G-structure; the structure is integrable if and only if $\Gamma = 0$. At a fixed point, Γ is an element of the G-representation $\mathbb{R}^n \otimes \mathfrak{m}$. Moreover, one can show that in any case when G is the isotropy group of some tensor $\mathcal{T}$ the algebraic G-types of Γ correspond to algebraic G-types of $\nabla^g \mathcal{T}$ (see [F03]; we also recommend [Agr06] as a suitable review on characteristic connections). We are looking again for metric connections with skew torsion and preserving the fixed G-structure. If it exists, it is called the characteristic torsion of the fixed G-type and we denote by T^c. However, not all G-structures admit such a connection. The question whether or not a certain G-type admits a characteristic connection can be decided using representation theory. Indeed, consider the G-morphism

$$\Theta : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m} , \quad \Theta(T) := \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner T) .$$

Theorem 1 ([F03]). *A G-structure of a Riemannian manifold admits a characteristic connection if and only if the intrinsic torsion Γ belongs to the image of Θ. In this case the characteristic torsion T and the intrinsic torsion are related by the formula $2 \cdot \Gamma = -\Theta(T)$.*

6. GRAY-HERVELLA CLASSIFICATION

Let (M, g, J) be 6-dimensional almost Hermitian manifold. Then the corresponding $U(3)$-structure is given by the Lie algebra decomposition $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}^6$. One can directly compute the decomposition of $\mathbb{R}^6 \otimes \mathfrak{m}^6$. The U(3)-representation $\mathbb{R}^6 \otimes \mathfrak{m}^6$ splits into four irreducible representations,

$$\mathbb{R}^6 \otimes \mathfrak{m}^6 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 .$$

These are the basic classes of $U(3)$-structures in the Gray-Hervella classification. The manifolds of type $\mathcal{W}_1$ are exactly the nearly Kähler manifolds. On the other side, the U(3)-representation $\Lambda^3(\mathbb{R}^6)$ splits into three irreducible components,

$$\Lambda^3(\mathbb{R}^6) = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 .$$

The reader can find an explicit description of these decompositions in the paper [AFS05]. Together, this allows us to describe more explicitly the U(3)-structures admitting a characteristic connection:

Corollary 6.1 ([FI02]). *A* U(3)-*structure admits a characteristic connection if and only if the* $\mathcal{W}_2$-*component of the intrinsic torsion vanishes.*

Let us finally summarize some results of Gray and Kirichenko on nearly Kähler manifolds in dimension 6. A nearly Kähler manifold is said to be of constant type if there exists a positive constant α such that for all vector fields

$$\|(\nabla_X^g J)(Y)\|^2 = \alpha[\|X\|^2 \|Y\|^2 - g(X,Y)^2 - g(JX,Y)^2].$$

Theorem 2 ([G76]). *Let* (M, g, J) *be a 6–dimensional nearly Kähler manifold that is not Kähler. Then*

    ii) *M is of constant type,*

    iii) *g is an Einstein metric on M,*

    iv) *the first Chern class of M vanishes.*

Theorem 3 ([Ki77]). *The characteristic torsion of a nearly Kähler 6-manifold is parallel with respect to the characteristic connection,* $\nabla^c T^c = 0$.

7. Spinorial approach

There is another characterization of 6-dimensional nearly Kähler manifolds due to R. Grunewald, involving the existence of so called real Killing spinors ([Gru90], see also [BFGK91]). Let us first introduce basic facts and definitions. The real Clifford algebra in dimensions 6 is isomorphic to $\mathrm{End}(\mathbb{R}^8)$. The spin representations is real, 8-dimensional and we denote it by $\Delta := \mathbb{R}^8$. By fixing an orthonormal basis $e_1, \ldots, e_6$ of the Euclidean space $\mathbb{R}^6$, one choice for the real representation of the Clifford algebra on Δ is

$$\begin{aligned}
e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}, \\
e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48}, \\
e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67},
\end{aligned}$$

where the matrices E_{ij} denote the standard basis elements of the Lie algebra $\mathfrak{so}(8)$, i.e. the endomorphisms mapping e_i to e_j, e_j to $-e_i$ and everything else to zero. The spin representation admits a $Spin(6)$-invariant complex structure $J : \Delta \to \Delta$ defined be the formula

$$J := e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot e_5 \cdot e_6.$$

Indeed, $J^2 = -1$ and J anti-commutes with the Clifford multiplication $X \cdot \phi$ by vectors $X \in \mathbb{R}^6$ and spinors $\phi \in \Delta$; this reflects the fact that Spin(6) is isomorphic to SU(4). The complexification of Δ splits,

$$\Delta \otimes_{\mathbb{R}} \mathbb{C} = \Delta^+ \oplus \Delta^-,$$

a consequence of the fact that J is a real structure making (Δ, J) complex-(anti)-linearly isomorphic to either $\Delta^{\pm}$, via $\phi \to \phi \pm i \cdot J(\phi)$. Furthermore, any real spinor $0 \neq \phi \in \Delta$ decomposes Δ into three pieces,

$$\Delta = \mathbb{R}\phi \oplus \mathbb{R} J(\phi) \oplus \{X \cdot \phi : X \in \mathbb{R}^6\}.$$

In particular, J preserves the subspaces $\{X \cdot \phi : X \in \mathbb{R}^6\} \subset \Delta$, and the formula

$$J_\phi(X) \cdot \phi := J(X \cdot \phi)$$

defines an orthogonal complex structure J_ϕ on $\mathbb{R}^6$ that depends on the spinor ϕ. Moreover, the spinor determines a 3-form by means of

$$\omega_\phi(X, Y, Z) := -(X \cdot Y \cdot Z \cdot \phi, \phi)$$

where the brackets indicate the inner product on Δ. The pair (J_ϕ, ω_ϕ) is an SU(3)-structure on $\mathbb{R}^6$, and any such arises in this fashion from some real spinor. All this can be summarized in the known fact that SU(3)-structures on $\mathbb{R}^6$ correspond one-to-one with real spinors of length one $(mod\, \mathbb{Z}_2)$,

$$\mathrm{SO}(6)/\mathrm{SU}(3) \;=\; \mathbb{P}(\Delta) \;=\; \mathbb{RP}^7 \,.$$

These formulas proves the following

Proposition 7.1. *Let M be a simply connected Riemannian spin manifold. Then the* SU(3)-*structures on M correspond to the real spinor fields of length one defined on M.*

The different types of SU(3)-structures in the sense of Gray-Hervella can be characterized by certain spinoral field equation for the defining spinor ϕ. The first result of this type has been obtained by R. Grunewald in 1990. A spinor field ϕ defined on a Riemannian spin manifold is called a real Killing spinor if it satisfies the following first order differential equation

$$\nabla^g_X \phi \;=\; \lambda \cdot X \cdot \phi, \quad \lambda = \mathrm{const} \in \mathbb{R} \,.$$

If $\lambda = 0$, the spinor field is simply parallel. Real Killing spinors are the eigenspinors of the Dirac operator realizing the lower bound of the Dirac spectrum given by Th. Friedrich in 1980, see [F80]. Now we can formulate the mentioned result:

Theorem 4. *(see* [Gru90]*) Let (M, g) be a 6-dimensional spin manifold admitting a non-trivial real Killing spinor. Then M is nearly Kähler. Conversely, any simply connected nearly Kähler 6-manifold admits non-trivial Killing spinor.*

We sketch the proof of the first statement. Suppose that ϕ is a Killing spinor, $\nabla^g_X \phi = X \cdot \phi$. We differentiate the equation $J_\phi(X) \cdot \phi = J(X \cdot \phi)$:

$$\nabla^g_Y(J_\phi(X)) \cdot \phi + J_\phi(X) \cdot \nabla^g_Y \phi \;=\; J(\nabla^g_Y X \cdot \phi) + J(X \cdot \nabla^g_Y \phi) \;=\; J_\phi(\nabla^g_Y X) \cdot \phi + J(X \cdot \nabla^g_Y \phi) \,.$$

This formula yield the derivative $\nabla^g J_\phi$:

$$(\nabla^g_Y J_\phi)(X) \cdot \phi \;=\; J(X \cdot \nabla^g_Y \phi) - J_\phi(X) \cdot \nabla^g_Y \phi \;=\; J(X \cdot Y \cdot \phi) - J_\phi(X) \cdot (Y \cdot \phi) \,.$$

In particular, for $X = Y$ we obtain

$$\begin{aligned}
(\nabla^g_X J_\phi)(X) \cdot \phi &= -\|X\|^2 J(\phi) - J_\phi(X) \cdot X \cdot \phi \;=\; -\|X\|^2 J(\phi) + X \cdot J_\phi(X) \cdot \phi \\
&= -\|X\|^2 J(\phi) + X \cdot J(X \cdot \phi) \;=\; -\|X\|^2 J(\phi) - X \cdot X \cdot J(\phi) \;=\; 0 \,.
\end{aligned}$$

Finally, the almost complex structure J_ϕ is nearly Kähler.

Remark 7.1. The spinor field equations for all other types of SU(3)-structures have been discussed in the paper [ACFH15] .

Example 7.1. The 6-dimensional sphere admits real Killing spinors. Indeed, fix a constant spinor in the Euclidean space $\mathbb{R}^7$ and restrict it to S^6. Then it becomes a real Killing spinor on the sphere. Moreover, this spinor defines its standard nearly Kähler structure described before.

8. Non-homogeneous nearly Kähler manifolds

Although it had been widely believed that non-homogeneous nearly Kähler manifolds should exist, their explicit construction was an open problem for many years–their odd-dimensional siblings, nearly parallel G_2-manifolds, had been much less reluctant to provide inhomogeneous examples. On the path to a solution, several approaches had been tried that provided new insights into the shape and properties of nearly Kähler manifolds, but had not brought the answer to the original problem. For example, nearly hypo structures allow the construction of compact nearly Kähler structures with conical singularitie [FIMU08], and infinitesimal deformations of nearly Kähler structures lead to interesting spectral problems on Laplacians [MS11]. Local homogeneous non-homogeneous examples of nearly Kähler manifolds were described in [CV15].

The main breakthrough was obtained very recently by Foscolo and Haskins [FH17], which we shall now shortly describe as it relates directly to our object of investigation, S^6:

Theorem 5 (Foscolo, Haskins). *There exists a non-homogeneous nearly Kähler structure on S^6 and on $S^3 \times S^3$.*

These are the first example of non-homogeneous compact nearly Kähler 6-manifolds. Recall that Butruille [B05] showed that the only homogeneous compact nearly Kähler 6-manifolds are S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold F_2. The examples of L. Foscolo and M. Haskins are based on weakening of the assumption of homogeneity: they are cohomogeneity one, i.e., they admit an isometric action of a compact Lie group such that generic orbits of the action are of codimension one. The Lie group considered in this case is $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and the generic orbits are $S^2 \times S^3$ which is motivated by results of Podesta and Spiro [PS12] characterizing all possible groups and orbits for cohomogeneity one nearly Kähler. In fact L. Foscolo and M. Haskins state the following conjecture.

Conjecture 1. The only simply connected cohomogeneity one compact nearly Kähler manifolds in dimension 6 are the spaces found in [FH17] on S^6 and $S^3 \times S^3$.

For proof of Theorem 5 they use another, equivalent (see for example [R93]) description of nearly Kähler 6-manifolds.

Proposition 8.1. *A 6-dimensional manifold (M, g, J) is nearly Kähler if and only if there exists a three holomorphic form $\omega \in \Lambda^{3,0}$ and a constant a such that the following conditions hold*

$$d\Omega = 12\, a\, \mathrm{Re}(\omega)\,, \quad d\mathrm{Im}(\omega) = a\, \Omega \wedge \Omega\,,$$

where $\Omega = g(J\cdot, \cdot)$ is the Kähler form.

This approach can be used to make explicit relation between nearly Kähler 6-manifolds and manifolds with G_2 holonomy which could have been suggested by the construction of the structure on S^6 from imaginary octonions. To see this, consider a 7-dimensional Riemannian cone $\mathcal{C}(M)$ over a smooth compact 6-manifold M and assume that the holonomy of $\mathcal{C}(M)$ is contained in G_2. Then, $\mathcal{C}(M)$ is equipped with a G_2 structure, i.e., a 3-form φ and its Hodge dual $*\varphi$ with special properties. On the level 1 of the cone (which can be identified with M) φ and $*\varphi$ induce $\mathrm{SU}(3)$ structure (ω, Ω) satisfying nearly Kähler conditions from Proposition 8.1.

The main idea of the proof is to consider so-called nearly hypo structures which are the SU(2) structures induced on oriented hypersurfaces of nearly Kähler 6-manifolds from SU(3)-structures. They describe the space of nearly hypo structures on $S^2 \times S^3$ invariant under SU(2) × SU(2) action showing that it is a smooth connected 4-manifold. Away from singular orbits, cohomogeneity one nearly Kähler manifolds correspond to curves on this space satisfying some ODE equations. It turns out that there is a 2-parameter family of solutions of the ODE, and to finish the proof they found conditions under which the solutions extend to compact nearly Kähler 6-manifold. It is important to note that this is closely related with desingularizations of Calabi-Yau conifold.

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