JEAN-PIERRE BOURGUIGNON, OUSSAMA HIJAZI, JEAN-LOUIS MILHORAT, ANDREI MOROIANU AND SERGIU MOROIANU, "A SPINORIAL APPROACH TO RIEMANNIAN AND CONFORMAL GEOMETRY"

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In 1928, P. A. M. Dirac introduced the differential equation for the state function of a particle with spin $\frac{1}{2}$, see [6]. He argued as follows: Consider a free classical particle in $\mathbb{R}^3$ obeying the laws of special relativity. Its mass m, energy E and momentum $p = \frac{vm}{\sqrt{1-v^2/c^2}}$ satisfy the well-known relation

$$E \;=\; \sqrt{c^2p^2 \,+\, m^2c^4}.$$

Quantizing the particle, the energy as well as the momentum are to be replaced by the differential operators

$$E \;\longrightarrow ih\,\frac{\partial}{\partial t}\;,\; p \;\longrightarrow -ih\,\mathrm{grad},$$

respectively. The state function ψ of the particle is thus a solution of the equation

$$ih\,\frac{\partial \psi}{\partial t} \;=\; \sqrt{c^2h^2\Delta \,+\, m^2c^4}\,\psi$$

involving the 3-dimensional Laplacian Δ, and leading to the question how this square root should be understood. Let us look for a square root $D = \sqrt{\Delta}$ of the Laplacian $\Delta = -\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ in arbitrary dimension. The assumption that D should be a first order differential operator with constant coefficients leads to the ansatz

$$D \;=\; \sum_{i=1}^{n}\gamma_i\frac{\partial}{\partial x_i},$$

where the coefficients γ_i satisfy the conditions

$$(*) \qquad \gamma_i\gamma_j \,+\, \gamma_j\gamma_i \;=\; -\,2\;\delta_{ij}.$$

Consequently, the state function ψ has to be a vector valued function, with values in a complex vector space admitting linear transformations γ_i with property $(*)$. This is the so called space of *spinors*, of dimension $2^{[n/2]}$. However, this space is not a representation of the group $\mathrm{SO}(n)$, but only of its universal covering $\mathrm{Spin}(n)$. Locally this is not a problem, but globally it has remarkable consequences.

In 1932 E. Schrödinger studied locally "das Diracsche Elektron" on semi-Riemannian manifolds [20]. In particular he compared the square of the Dirac operator D with the

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Laplacian and observed that there is a difference term depending on the scalar curvature R of the manifold,

$$D^2 \;=\; \Delta + \frac{1}{4}\,R.$$

Interestingly, E. Schrödinger was very well aware that he had obtained an important formula. His paper ends with the following comment:

Das zweite Glied scheint mir von erheblichem theoretischen Interesse. Es ist freilich um viele, viele Zehnerpotenzen zu klein, um etwa das Glied rechter Hand ersetzen zu können. Denn μ ist die reziproke Compton-Wellenlänge, ungefähr $10^{11} cm^{-1}$. Immerhin scheint es bedeutungsvoll, dass in der verallgemeinerten Theorie überhaupt ein mit dem rätselhaften Masseglied gleichartiges ganz von selber angetroffen wird.

A Dirac operator cannot be defined globally on any oriented Riemannian manifold M^n. Indeed, one needs a complex vector bundle equipped with endomorphisms γ_i $(1 \leq i \leq n)$ satisfying the algebraic relations $(*)$. This restricts the topological type of the manifold, the first and second Stiefel-Whitney classes have to vanish (so called *spin manifolds*). For example, all odd-dimensional complex projective spaces are spin, whereas the even-dimensional ones are not spin. At the Arbeitstagung in 1962, M. F. Atiyah introduced mathematically rigorously the Dirac operator as a first order elliptic operator for Riemannian spin manifolds and discussed the index [2]. Since then it has become one of the basic elliptic operators in analysis, geometry, representation theory and topology.

Shortly after, A. Lichnerowicz used the Dirac operator together with the general index formula for the proof that the $\hat{\mathcal{A}}$-genus of a compact Riemannian spin manifold of dimension divisible by 4 and with positive scalar curvature vanishes [17]. This has been the first known obstruction to the existence of metrics with positive scalar curvature. A. Lichnerowicz – not being aware of the result of Schrödinger – computed once again the square of the Dirac operator. If the scalar curvature is positive, there are no harmonic spinors, i. e. the index is zero. N. Hitchin generalised this result to any dimension [13]. Moreover, he explained many properties of the Dirac operator depending on the underlying Riemannian metric and he computed some spectra explicitly. In particular "he discovered that, in contrast with the Laplacian on exterior forms, the dimension of the space of harmonic spinors is a conformal invariant which can (dramatically) change with the metric" (page 5 of the reviewed book).

Since the beginning of the 70ties the Dirac operator plays an important role in representation theory, see [21], [19]. On a Riemannian symmetric space the Dirac operator is an invariant differential operator and one can compare its square D^2 with the Casimir operator Ω. One obtains a new formula different from the Schrödinger-Lichnerowicz formula,

$$D^2 \;=\; \Omega + \frac{1}{8}\,R.$$

This so-called Parthasarathy formula yields an effective method for the computation of the Dirac spectrum of compact Riemannian symmetric spaces.

The Schödinger-Lichnerowicz formula bounds the eigenvalues λ of the Dirac operator of a compact spin manifold by $\lambda^2 \geq R_{\min}/4$, where $R_{\min}$ denotes the minimum of the scalar curvature. In 1980 Th. Friedrich observed that this estimate is never optimal in case $R_{\min} > 0$. Indeed, the optimal inequality reads [7]

$$\lambda^2 \;\geq\; \frac{n}{4(n-1)}\, R_{\min}.$$

On spheres this lower bound is really an eigenvalue of D^2. In the mentioned paper an example is given of an Einstein metric on the 5-dimensional Stiefel manifold $SO(4)/SO(2)$ for which the lower bound is again a Dirac eigenvalue, i. e. an Obata-type theorem known for Laplacians does not hold for this Dirac estimate. Instead, one proves that if the lower bound is an eigenvalues of D^2, then the space must be Einstein and the eigenspinor ψ satisfies the *real Killing spinor* equation [7],

$$\nabla_X \psi \;=\; \frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\, X \cdot \psi,$$

where $X \cdot \psi$ denotes the Clifford multiplication of the spinor ψ by the vector X. Spaces with real Killing spinors and their link to special geometric structures have been investigated in dimensions $4 \leq n \leq 8$ by Friedrich/Kath, Grunewald, see [9], [10], [11], [14]. Sasaki-Einstein manifolds in all odd dimensions admit Killing spinors, see Friedrich / Kath [10]. Unfortunately, the authors of the present book didn't include these articles in their list of references nor in their exposition, thus giving an incomplete picture in Section 8.3. Some years later C. Bär proved in higher dimensions $n \geq 9$ the converse: except spheres in all dimensions, only Sasaki-Einstein manifolds admit real Killing spinors [3].

Following an invitation of Jean-Pierre Bourguignon, I visited the École Polytechnique in spring 1984. At that time Oussama Hijazi was his PhD student. Both were looking for a conformal estimate for Dirac operators in order to get a refinement of the lower Dirac bound [12]. The estimate depends on the lowest eigenvalue $\mu_1(M^n, g)$ of the conformal Laplacian (Yamabe operator),

$$\lambda^2 \;\geq\; \frac{n}{4(n-1)}\, \mu_1(M^n, g).$$

Moreover, they observed that a compact Riemannian manifold with a Killing spinor cannot admit parallel forms (Theorem 5.17 of the present book). In particular, this implies for Kähler manifolds that the previous lower bound can never be an eigenvalue of the Dirac operator. The optimal lower bound for Kähler manifolds was proved by K.-D. Kirchberg [15, 16]. Ten years later, Kramer / Semmelmann / Weingart obtained the optimal lower Dirac bound for quaternionic Kähler manifolds.

A. Lichnerowicz added a new idea to the discussed topic in 1987 [18]. He considered the second universal first order differential operator acting on spinors, the so called *twistor operator*. Its kernel is a conformal invariant and consist of all spinor field ψ satisfying the differential equation

$$\nabla_X \psi \;+\; \frac{1}{n} X \cdot D\psi \;=\; 0.$$

Real and imaginary Killing spinors are special solutions of the twistor equation. In [18] and [8] the authors studied the solutions of the twistor equation in more details. In particular, such a spinor field vanishes only in isolated points, and outside this discrete set the twistor spinor is conformally equivalent to a Killing spinor or a parallel spinor. Complete Riemannian manifolds with imaginary Killing spinors are warped products of a manifold with parallel spinors and $\mathbb{R}$, see [4], [5].

This is the background—roughly until 1990—of the present book on Dirac operators in geometry. Most of the results described above are treated in it. There are several other textbooks on the subject, for example Baum / Friedrich / Grunewald / Kath 1991, Friedrich 2000 and Ginoux 2009. During the last 25 years, the number of publications in the topic increased drastically and the new book contains some of these further results.

The first part "Basic spinorial material" of the book introduces a beginner into Clifford algebras, representations of the spin group, spin structures on manifolds, properties and examples of Dirac and twistor operators. The reader finds here many basic formulas with complete proofs, for example the Schrödinger-Lichnerowicz formula. There is a section on pseudo-differential operators on compact manifolds without boundary in general and on spectral properties of self-adjoint elliptic operators specifically. Alltogether, these 110 pages are a self-contained presentation of the basic algebraic, geometric and analytic ingredients needed for the study of Dirac operators. The material can be used for a basic course introducing into the spin geometry.

The second part of the book is devoted to lower eigenvalue estimates of the Dirac operator on closed spin manifolds. The already discussed inequalities of Friedrich (1980), Hijazi (1986), Kirchberg (1986), Kramer /Semmelmann /Weingart (1998) and Moroianu /Ornea (2004) are completely proved, sometimes with different arguments than in the original publications. Moreover, using the integral formulas in each case one can derive the first order differential equation for a spinor field being an eigenspinor with the lowest possible eigenvalue. These spinorial field equations are stronger then the eigenvalue equation (Killing spinors, Kählerian Killing spinors etc.) and the existence of such an extremal eigenspinor restricts the underlying geometry rather severely. The discussion of the integrability conditions for Killing spinors in the general Riemannian case, the Kähler case as well as the quaternionic-Kähler case is the contents of the third part "Special spinor fields and geometries" of the book. Finally one obtains a description of Riemannian manifolds with Killing spinors (see above), of Kähler manifolds with Kählerian Killing spinors / twistors (Kirchberg 1988 in complex dimension 3, Friedrich 1993 in complex dimension 2, Moroianu 1995 in higher dimensions) and last not least of the possible quaternionic-Kähler manifolds (Kramer / Semmelmann / Weingart 1998).

The last part "Dirac spectra of model spaces" of the book is of different flavour. The aim is the explicit computation of the Dirac spectrum for some compact Riemannian symmetric spaces. Using the Parthasarathy formula discussed above and the Peter-Weyl theorem for homogeneous vector bundles, the Dirac spectrum can be computed via the representation theory of the isometry group of the symmetric space. Consequently, the authors give a brief survey of the representations for some classical groups and compute finally the Dirac spectra on spheres and projective spaces.

The book is an interesting introduction into Dirac operators on compact Riemannian manifolds. It is self-contained and may serve as a guideline for everybody working in Differential Geometry or Mathematical Physics. All the Dirac operators discussed therein depend on torsion-free connections. Since 15 years Dirac operators, depending on more general metric connections with non-trivial torsion play an important role in Differential Geometry and Mathematical Physics. Many of the results contained in the present book have been discussed in this more general situation. These Dirac operators are used in order to understand non-integrable geometries and occur in string theory, see [1]. In other words, the topic of the present book is still an active area in

Differential Geometry and Mathematical Physics with many new results and interesting open questions.

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