COCALIBRATED G_2-MANIFOLDS WITH RICCI FLAT CHARACTERISTIC CONNECTION

THOMAS FRIEDRICH

ABSTRACT. Any 7-dimensional cocalibrated G_2-manifold admits a unique connection ∇ with skew symmetric torsion (see [8]). We study these manifolds under the additional condition that the ∇-Ricci tensor vanish. In particular we describe their geometry in case of a maximal number of ∇-parallel vector fields.

1. Introduction

Consider a triple $(M^n\,,\,g\,,\,\mathrm{T})$ consisting of a Riemannian manifold $(M^n\,,\,g)$ equipped with a 3-form T. We denote by ∇^g , Ric^g and Scal^g the Levi-Civita connection, the Riemannian Ricci tensor and the scalar curvature. The formula

$$\nabla_X Y \ := \ \nabla^g_X Y \ + \ \frac{1}{2}\,\mathrm{T}(X\,,\,Y\,,\,-)$$

defines a metric connection with torsion T. We will denote by Ric^∇ and Scal^∇ its Ricci tensor and scalar curvature respectively. If the Ricci tensor $\mathrm{Ric}^\nabla = 0$ vanishes, then T is a coclosed form, $\delta\mathrm{T} = 0$, and the Riemannian Ricci tensor is completely given by the 3-form T (see [8]),

$$\mathrm{Ric}^g(X\,,\,Y) \ = \ \frac{1}{4}\sum_{i,j=1}^{n}\mathrm{T}(X\,,\,e_i\,,\,e_j)\cdot\mathrm{T}(Y\,,\,e_i\,,\,e_j)\ , \quad \mathrm{Scal}^g \ = \ \frac{3}{2}\,\|\mathrm{T}\|^2\ .$$

In particular, the Ricci tensor Ric^g is non-negative, $\mathrm{Ric}(X,X) \geq 0$.

Let us introduce the 4-form σ_{T} depending on T,

$$\sigma_{\mathrm{T}} \ = \ \frac{1}{2}\sum_{i=1}^{n}(e_i \,\lrcorner\, \mathrm{T}) \wedge (e_i \,\lrcorner\, \mathrm{T})\ .$$

If moreover there exists a ∇-parallel spinor field Ψ, then there is an algebraic link between $d\mathrm{T}$, $\nabla\mathrm{T}$ and σ_{T} (see [8]),

$$\big(X \,\lrcorner\, d\mathrm{T} \ + \ 2\,\nabla_X\mathrm{T}\big)\cdot\Psi \ = \ 0\ , \quad \big(3\,d\mathrm{T} \ - \ 2\sigma_{\mathrm{T}}\big)\cdot\Psi \ = \ 0\ .$$

The classification of flat metric connections with skew symmetric torsion has been investigated by Cartan and Schouten in 1926. Complete proofs are known since the beginning of the 70-ties. In [4] one finds a simple proof of this result. Therefore, we are interested in non-flat ($\mathcal{R}^\nabla \not\equiv 0$) and ∇-Ricci flat ($\mathrm{Ric}^\nabla \equiv 0$) metric connections with skew symmetric torsion $\mathrm{T} \not\equiv 0$.

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In this paper we study the 7-dimensional case. Any cocalibrated G_2-manifold admits a unique connection ∇ with skew symmetric torsion and ∇-parallel spinor field Ψ. If this characteristic connection is Ricci flat, then we obtain a solution of the Strominger equations (see [8]),

$$\nabla \Psi \ = \ 0, \quad \mathrm{Ric}^{\nabla} \ = \ 0, \quad d * \mathrm{T} \ = \ 0.$$

If $\mathrm{T} = 0$, M^7 is a Riemannian manifold with holonomy G_2 and $\mathrm{Ric}^g = 0$ follows automatically. The case of $\mathrm{T} \not\equiv 0$ is different. The condition $\mathrm{Ric}^{\nabla} \equiv 0$ is not a consequence of the fact that the holonomy of ∇ is contained in G_2, it is a new condition for the cocalibrated G_2-structure. In this paper we investigate the geometry of the 7-manifolds under consideration. Moreover, we describe all these manifolds with a large number of ∇-parallel vector fields.

2. **Examples of Ricci flat connections with skew symmetric torsion**

Let us discuss some examples.

Example 2.1. Any Hermitian manifold admits a unique metric connection ∇ preserving the complex structure and with skew symmetric torsion (see [8]) . In [10] the authors constructed on $(k-1)\,(S^2 \times S^4) \,\#\, k\,(S^3 \times S^3)$ a Hermitian structure with vanishing ∇-Ricci tensor, $\mathrm{Ric}^{\nabla} = 0$, for any $k \geq 1$. These examples are toric bundles over special Kähler 4-manifolds.

Example 2.2. There are 7-dimensional cocalibrated G_2-manifolds $(M^7\,,\,g\,,\,\omega^3)$ with characteristic torsion T such that

$$\nabla \mathrm{T} \ = \ 0\ , \quad d\mathrm{T} \ = \ 0\ , \quad \delta \mathrm{T} \ = \ 0\ , \quad \mathrm{Ric}^{\nabla} \ = \ 0\ , \quad \mathfrak{hol}(\nabla) \ \subset \ \mathfrak{u}(2) \ \subset \ \mathfrak{g}_2\ .$$

The regular G_2-manifolds of this type have been described in [7], Theorem 5.2 (the degenerate case $2a + c = 0$). M^7 is the product $X^4 \times S^3$, where X^4 is a Ricci-flat Kähler manifold and S^3 the round sphere.

Example 2.3. A suitable deformation of any Sasaki-Einstein manifold yields a metric connection with skew symmetric torsion and vanishing Ricci tensor, see [1].

Next we describe a similar method in order to construct 5-dimensional connections with skew symmetric torsion and vanishing Ricci tensor.

Theorem 2.1. *Let $(Z^4\,,\,g\,,\,\Omega^2)$ be a 4-dimensional Riemannian manifold equipped with a 2-form Ω^2 such that*

1. $d\Omega^2 \ = \ 0$, $d * \Omega^2 \ = \ 0$ *and* $\Omega^2 \wedge \Omega^2 \ = \ 0$.
2. *The 2-dimensional distributions*
$$E^2 \ = \ \big\{X \in TZ^4\ :\ X \,\lrcorner\, \Omega^2 \ = \ 0\big\}\,, \quad F^2 \ = \ \big\{X \in TZ^4\ : X \perp E^2\big\}$$
are integrable.
3. *The 2-form is of the form $\Omega^2 = 2a\, f_1 \wedge f_2$, where a is constant and f_1, f_2 is an oriented orthonormal frame in F^2.*
4. *The Riemannian Ricci tensor of Z^4 has two non-negative eigenvalues of multiplicity two,*
$$\mathrm{Ric}^g \ = \ 4a^2\,\mathrm{Id} \text{ on } F^2, \quad \mathrm{Ric}^g \ = \ 0 \text{ on } E^2.$$
5. Ω^2 *is the curvature form of some $\mathbb{R}^1$- or S^1-connection η.*

Then the principal fiber bundle $\pi : N^5 \to Z^4$ defined by Ω^2 admits a Riemannian metric and the torsion form

$$\mathrm{T} \;=\; \pi^*(\Omega^2) \wedge \eta$$

yields a metric connection ∇ with the following properties:

$$||\mathrm{T}||^2 \;=\; 4a^2 \;,\; d\mathrm{T} \;=\; 0 \;,\; d*\mathrm{T} \;=\; 0 \;,\; \mathrm{Ric}^{\nabla} \;=0 \;,\; \nabla\eta \;=\; 0 \;.$$

Proof. Apply O'Neill's formulas and compute

$$\mathrm{Ric}^g(X\,,\,Y) \;-\; \frac{1}{4}\sum_{i,j=1}^{5} \mathrm{T}(X\,,\,e_i\,,\,e_j)\cdot\mathrm{T}(Y\,,\,e_i\,,\,e_j) \;=\; 0\;. \qquad \square$$

Example 2.4. Let $u = u(x, y)$ be a smooth function of two variables and consider the metric

$$g \;=\; e^u\, x\left(dx^2 \,+\, dy^2\right) \;+\; x\,dz^2 \;+\; \frac{1}{x}\left(dt \,+\, y\,dz\right)^2$$

defined on the set $Z^4 = \{(x, y, t, z) \in \mathbb{R}^4 : x > 0\}$. (Z^4, g) is a Kähler manifold and the Riemannian Ricci tensor has two eigenvalues, namely zero and

$$-\frac{u_{xx} + u_{yy}}{2xe^u},$$

both with multiplicity two (see [5] , [11]). If the function u is a solution of the equation

$$-\frac{u_{xx} + u_{yy}}{2xe^u} \;=\; 4a^2,$$

Theorem 2.1 is applicable and we obtain a family of non-flat 5-dimensional examples. Remark that a compact Kähler manifold Z^4 of that type splits into $S^2 \times T^2$, see [6]. The corresponding connection ∇ on the Lie group $N^5 = S^3 \times T^2$ is flat, see [4] .

3. Cocalibrated G_2-manifolds with vanishing characteristic Ricci tensor

Consider a cocalibrated G_2-manifold $(M^7\,, g\,,\,\omega^3)$,

$$d*\omega^3 \;=\; 0, \quad \|\,\omega^3\,\|^2 \;=\; 7,$$

and suppose that the G_2-structure ω^3 is not ∇^g-parallel (i.e. $d\,\omega^3 \not\equiv 0$). There exists a unique metric connection ∇ with skew symmetric torsion and preserving the G_2-structure ω^3. Its torsion form is given by the formula (see [8]),

$$\mathrm{T} \;=\; -\,*\,d\omega^3 \;+\; \mu\,\omega^3\;, \quad \mu \;=\; \frac{1}{6}\left(d\omega^3\,,\,*\omega^3\right)\,.$$

The condition $\mathrm{Ric}^{\nabla} = 0$ becomes equivalent to $d\,\mathrm{T} = 0$ and $d*\mathrm{T} = 0$. Indeed, we have:

Theorem 3.1 ([8, Thm 5.4])**.** *The following conditions are equivalent:*

1. $\mathrm{Ric}^{\nabla} \,=\, 0$.
2. $d\,\mathrm{T} \,=\, 0$ *and* $d*\mathrm{T} \,=\, 0$.
3. $d\mu \,=\, 0$ *and* $d*d\omega^3 \,-\, \mu\,d\,\omega^3 \,=\, 0$.

Using the G_2-splitting of 3-forms, $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, we know that the characteristic torsion of a cocalibrated G_2-manifold belongs to $\mathrm{T} \in \Lambda^3_1 \oplus \Lambda^3_{27}$. In particular, we obtain

$$\mathrm{T} \wedge \omega^3 \;=\; 0\;.$$

Differentiating the latter equation and using $d\,\mathrm{T} = 0$ one gets

$$\left(*\;d\,\omega^3 - \mu\,\omega^3\right)\wedge\omega^3 \;=\; 0, \quad \|d\,\omega^3\|^2 \;=\; 6\,\mu^2.$$

We compute the length of T,

$$\|\mathrm{T}\|^2 \;=\; \|d\omega^3\|^2 - 2\,\mu\big(\,*\,d\omega^3,\,\omega^3\big) + 7\,\|\omega^3\|^2 \;=\; 6\,\mu^2 - 12\,\mu^2 + 7\mu^2 \;=\; \mu^2 .$$

Consequently, $\|\mathrm{T}\|^2$ is constant. Moreover, the Riemannian scalar curvature is constant, too,

$$\mathrm{Scal}^g \;=\; \frac{3}{2}\,\|\mathrm{T}\|^2 \;=\; \frac{3}{2}\,\mu^2 .$$

Since $(\mathrm{T},\omega^3) \;=\; \mu$, we decompose the torsion form into two parts according to the splitting of 3-forms,

$$\mathrm{T} \;=\; \mathrm{T}_1 + \mathrm{T}_{27}, \quad \mathrm{T}_1 \;=\; \frac{1}{7}\,\mu\,\omega^3, \quad \mathrm{T}_{27} \;=\; -\,*\,d\omega^3 + \frac{6}{7}\mu\omega^3 .$$

Corollary 3.1 ([8, Remark 5.5]). *Let $(M^7\,,\,g\,,\,\omega^3)$ be a compact, cocalibrated* G_2*-manifold with* $\mathrm{Ric}^\nabla = 0$ *and* $\mathrm{T} \neq 0$*. Then the third cohomology group is non-trivial,*

$$H^3(M^7\,;\,\mathbb{R}) \;\neq\; 0\ .$$

Example 3.1. On the round sphere S^7 there exists a G_2-structure (not cocalibrated) such that $\mathcal{R}^\nabla = 0$ (see [4]). In particular, the Ricci tensor vanishes, $\mathrm{Ric}^\nabla = 0$. The characteristic torsion is coclosed, $\delta\mathrm{T} = 0$, but not closed, $d\,\mathrm{T} \neq 0$.

Remark 3.1. A cocalibrated G_2-manifold with $\mathrm{Ric}^\nabla = 0$ and $\mathrm{T} \not\equiv 0$ cannot be of pure type Λ^3_1 or Λ^3_{27}. Indeed, if

$$0 \;=\; \mathrm{T}_{27} \;=\; -\,*\,d\omega^3 + \frac{6}{7}\mu\omega^3$$

we differentiate,

$$0 \;=\; -\,d\,*\,d\,\omega^3 \;+\; \frac{6}{7}\,\mu\,d\,\omega^3$$

and combine the latter formula with equation (3) of Theorem 3.1. We conclude that $\mu = 0$, $d\omega^3 = 0$ and, finally, $\mathrm{T} = 0$. The second case, i. e. $\mathrm{T}_1 = 0$, implies immediately $\mu = 0$ and $\mathrm{T} = 0$.

There exists a canonical ∇-parallel spinor field Ψ_0 such that

$$\nabla\Psi_0 \;=\; 0\ , \quad \omega^3\cdot\Psi_0 \;=\; -\,7\,\Psi_0\ .$$

Since $\Lambda^3_{27}\cdot\Psi_0 = 0$ we obtain

$$\mathrm{T}\cdot\Psi_0 \;=\; \mathrm{T}_1\cdot\Psi_0 \;=\; -\,\mu\,\Psi_0\ .$$

The integrability condition for a parallel spinor (see [8]) yields an algebraic restriction for the derivative $\nabla\mathrm{T}$, namely

$$\nabla_X\big(\mathrm{T}\cdot\Psi\big) \;=\; \big(\nabla_X\mathrm{T}\big)\cdot\Psi \;=\; 0\ , \quad \sigma_{\mathrm{T}}\cdot\Psi \;=\; 0\ , \quad \mathrm{T}^2\cdot\Psi \;=\; \|\,\mathrm{T}\,\|^2\,\Psi$$

for any vector $X \in TM^7$ and any ∇-parallel spinor field Ψ. In particular, the characteristic torsion T acts on the space of all ∇-parallel spinors. This condition is not so restrictive. For example, the space of 3-forms $\Sigma^3 \in \Lambda^3_{27}$ killing three spinors has dimension 14, the space killing four spinors has still dimension 9.

4. ∇-parallel vector fields

Via the Riemannian metric we identify vectors with 1-forms. Denote by $\mathcal{P}^\nabla$ the space of all ∇-parallel vector field (1-forms). Any ∇-parallel vector field θ is a Killing field and

$$2\,\nabla^g\theta \;=\; d\,\theta \;=\; \theta \lrcorner \mathrm{T}\ , \quad \nabla^g_\theta\theta \;=\; 0\ .$$

holds. This formula together with $d\,\mathrm{T} = 0$ implies that T is preserved by the flow of θ,

$$\mathcal{L}_\theta \mathrm{T} \;=\; 0\ .$$

The Riemannian Ricci tensor on θ becomes

$$\mathrm{Ric}^g(\theta\,,\,\theta) \;=\; \frac{1}{2}\,\|\,d\,\theta\,\|^2\ .$$

The subgroup of G_2 preserving four vectors in $\mathbb{R}^7$ is trivial. The isotropy subgroups of two or three vectors in $\mathbb{R}^7$ coincide and this group is isomorphic to $\mathrm{SU}(2) \subset G_2$. Finally, the isotropy subgroup of one vector is isomorphic to $\mathrm{SU}(3) \subset G_2$ (see for example [7]). This algebraic observation proves immediately the following

Proposition 4.1. *If (M^7, g, ω^3) is not ∇-flat, then the possible dimensions of the space $\mathcal{P}^\nabla$ are 0, 1, or 3.*

4.1. **The case of three ∇-parallel vector fields.** We discuss the case that there are three orthonormal and ∇-parallel 1-forms θ_1, θ_2, θ_3. Then $\omega^3(\theta_1,\,\theta_2,\,-)$ is ∇-parallel, too. If it does not coincide with θ_3 , then we have at least four ∇-parallel 1-forms, i.e. the G_2-connection ∇ is flat. Under our assumption $\mathcal{R}^\nabla \not\equiv 0$ we conclude that

$$\omega^3(\theta_1\,,\,\theta_2\,,\,-) \;=\; \theta_3\ , \quad \omega^3(\theta_1\,,\,\theta_2\,,\,\theta_3) \;=\; 1\ .$$

The holonomy of the connection ∇ is contained in $\mathfrak{su}(2) \subset \mathfrak{g}_2$. Moreover, the spinors

$$\Psi_0, \quad \Psi_1 \;:=\; \theta_1\cdot\Psi_0, \quad \Psi_2 \;:=\; \theta_2\cdot\Psi_0, \quad \Psi_3 \;:=\; \theta_3\cdot\Psi_0$$

are all ∇-parallel spinors. The torsion form T acts as a symmetric endomorphism on the space $\mathrm{Lin}(\Psi_0,\Psi_1,\Psi_2,\Psi_3)$ and $\mathrm{T}\cdot\Psi_0 = -\,\mu\ \Psi_0$. Consequently, T acts on the 3-dimensional space $\mathrm{Lin}(\Psi_1,\Psi_2,\Psi_3)$ and $\mathrm{T}^2 \;=\; \|\mathrm{T}\|^2\cdot\mathrm{Id} = \mu^2\cdot\mathrm{Id}$. We decompose the torsion form into

$$\mathrm{T} \;=\; \mathrm{T}_1 + \mathrm{T}_{27} \;=\; \frac{1}{7}\,\mu\,\omega^3 + \mathrm{T}_{27}$$

and we use the known action of ω^3 on spinors:

$$\omega^3\cdot\Psi_0 \;=\; -7\Psi_0, \quad \omega^3\cdot\Psi_i \;=\; \Psi_i,\ i=1,2,3, \quad \mathrm{T}_{27}\cdot\Psi_0 \;=\; 0.$$

Finally, $\mathrm{T}_{27} \in \Lambda^3_{27}$ preserves the space $\mathrm{Lin}(\Psi_1,\Psi_2,\Psi_3)$ and

$$\mathrm{T}^2_{27} + \frac{2}{7}\,\mu\,\mathrm{T}_{27} \;=\; \frac{48}{49}\,\mu^2.$$

Without loss of generality we may assume that Ψ_1, Ψ_2, Ψ_3 are eigenspinors of T_{27},

$$\mathrm{T}_{27}\cdot\Psi_i \;=\; m_i\,\Psi_i\ , \quad m_i^2 \;+\; \frac{2}{7}\,m_i\,\mu \;=\; \frac{48}{49}\,\mu^2,\ i \;=\; 1,2,3.$$

We fix an orthonormal basis $e_1\,, \ldots\,, e_7$ such that

$$\omega^3 \;=\; e_{127} \,+\, e_{135} \,-\, e_{146} \,-\, e_{236} \,-\, e_{245} \,+\, e_{347} \,+\, e_{567}$$

and $\theta_1 = e_1$, $\theta_2 = e_2$, $\theta_3 = e_7$. This is possible, since we already have $\omega^3(\theta_1, \theta_2, \theta_3) = 1$. Let

$$\mathrm{T}_{27} \;=\; \sum_{i<j<k} t_{ijk}\, e_{ijk}$$

be the 3-form T_{27} and introduce the following numbers:

$$a \;:=\; t_{236} + t_{245}\,, \quad b \;:=\; t_{347} + t_{567}, \quad c \;:=\; t_{235} - t_{246}.$$

A purely algebraic computation yields the following

Lemma 4.1. *The space of all 3-forms $\mathrm{T}_{27} \in \Lambda^3_{27}$ such that $\mathrm{T}_{27} \cdot \Psi_i = m_i\, \Psi_i$, $i = 1,2,3$ is an affine space of dimension 9. A parameterization is given by*

$$\begin{aligned} \mathrm{T}_{27} \;=\; & \big(-\frac{m_1}{2} - b\big)\, e_{127} - t_{156}\, e_{134} + \big(\frac{m_1}{2} + t_{146} + a\big)\, e_{135} \\ & - t_{145}\, e_{136} + t_{145}\, e_{145} + t_{146}\, e_{146} + t_{156}\, e_{156} - t_{256}\, e_{234} \\ & + t_{235}\, e_{235} + t_{236}\, e_{236} + t_{245}\, e_{245} + t_{246}\, e_{246} + t_{256}\, e_{256} + t_{347}\, e_{347} \\ & + t_{467}\, e_{357} - t_{457}\, e_{367} + t_{457}\, e_{457} + t_{467}\, e_{467} + t_{567}\, e_{567}\ . \end{aligned}$$

and

$$m_1 + 2a + 2b \;=\; m_2, \quad -2a + 2b \;=\; m_3, \quad c \;=\; 0.$$

Corollary 4.1. *For $X \perp \mathrm{Lin}(\theta_1, \theta_2, \theta_3)$ we have*

$$\mathrm{T}(\theta_i\,, \theta_j\,, X) \;=\; 0\;, \quad \mathrm{T} \;=\; (\theta_1 \lrcorner \mathrm{T}) \wedge \theta_1 \;+\; (\theta_2 \lrcorner \mathrm{T}) \wedge \theta_2 \;+\; (\theta_3 \lrcorner \mathrm{T}) \wedge \theta_3\ .$$

We solve the linear system with respect to a and b :

$$a \;=\; -\frac{1}{4}\big(m_1 - m_2 + m_3\big)\ , \quad b \;=\; \frac{1}{4}\big(- m_1 + m_2 + m_3\big)$$

In particular,

$$m_1 + 2\,b \;=\; \frac{1}{2}\big(m_1 + m_2 + m_3\big)\ .$$

We are interested in the value

$$\mathrm{T}(\theta_1, \theta_2, \theta_3) \;=\; \frac{1}{7}\,\mu - \frac{m_1}{2} - b \;=\; \frac{1}{7}\,\mu - \frac{1}{4}\big(m_1 + m_2 + m_3\big).$$

We have 8 possibilities, namely

$$m_i \;=\; \frac{6}{7}\,\mu\ , \quad \text{or} \quad m_i \;=\; -\frac{8}{7}\,\mu\ .$$

Therefore,

$$\mathrm{T}(\theta_1, \theta_2, \theta_3) \;=\; 0,\ \pm\frac{1}{2}\,\mu \quad \text{or} \quad \mu.$$

We summarize the result.

Theorem 4.1. *Let (M^7, g, ω^3) be a cocalibrated G_2-manifold and ∇ its characteristic connection. Suppose that $\mathrm{Ric}^{\nabla} = 0$, $\|\mathrm{T}\|^2 = \mu^2 > 0$ and $\mathcal{R}^{\nabla} \not\equiv 0$. If $\theta_1, \theta_2, \theta_3$ are three orthonormal and ∇-parallel vector fields, then*

1. $\omega^3(\theta_1, \theta_2, \theta_3) = 1$.
2. $\mathrm{T}(\theta_1, \theta_2, \theta_3)$ *is constant and has only four possible values,* 0, $\pm\mu/2$, μ.
3. $\mathrm{T}(\theta_i, \theta_j, X) = 0$ *for* $X \perp \mathrm{Lin}(\theta_1, \theta_2, \theta_3)$.

In particular

$$\begin{aligned} \mathrm{T} &= (\theta_1 \lrcorner \mathrm{T}) \wedge \theta_1 + (\theta_2 \lrcorner \mathrm{T}) \wedge \theta_2 + (\theta_3 \lrcorner \mathrm{T}) \wedge \theta_3 \\ &= d\,\theta_1 \wedge \theta_1 + d\theta_2 \wedge \theta_2 + d\theta_3 \wedge \theta_3 \ . \end{aligned}$$

and

$$[\theta_1, \theta_2] \ = \ -\mathrm{T}(\theta_1, \theta_2, \theta_3)\, \theta_3$$

is proportional to θ_3. *The* 3*-dimensional space* $\mathrm{Lin}(\theta_1, \theta_2, \theta_3)$ *is closed with respect to the Lie bracket and is a Lie subalgebra of the Killing vector fields. This algebra is either commutative or isomorphic to* $\mathfrak{so}(3)$.

Remark 4.1. Since we do not assume that the torsion form T is ∇-parallel, it is not obvious by general arguments that $[\theta_1\,,\,\theta_2] = -\,\mathrm{T}(\theta_1\,,\,\theta_2)$ is again ∇-parallel.

We can classify the case of $\mathrm{T}(\theta_1, \theta_2, \theta_3) = \mu$ immediately. Indeed, we have then $\|\mathrm{T}\|^2 \geq \mu^2$. On the other hand, we know that $\|\mathrm{T}\|^2 = \mu^2$ holds. It follows that

$$\mathrm{T} = \ \mu\,\theta_1 \wedge \theta_2 \wedge \theta_3 \quad \text{and} \quad \nabla \mathrm{T} \ = \ 0 \ .$$

Cocalibrated G$_2$-structures with characteristic holonomy $\mathfrak{su}(2)$ and a characteristic torsion of the given type have been classified at the end of our paper [7]. We apply this result and obtain

Theorem 4.2. *Let* (M^7, g, ω^3) *be a complete, cocalibrated* G$_2$*-manifold and* ∇ *its characteristic connection. Suppose that* $\mathrm{Ric}^\nabla = 0$. *If* $\theta_1, \theta_2, \theta_3$ *are three orthonormal and* ∇*-parallel vector fields and* $\mathrm{T}(\theta_1, \theta_2, \theta_3) = \mu$, *then the universal covering of* M^7 *is isometric to the product* $X^4 \times S^3$, *where* X^4 *is a complete anti-self dual and Ricci flat Riemannian manifold.*

If $\mathrm{T}(\theta_1, \theta_2, \theta_3) = 0$ the 3-dimensional abelian Lie group acts on M^7 locally free as a group of isometries and preserves the torsion form T. Moreover, we obtain the 2-forms $d\theta_i = \theta_i \lrcorner \mathrm{T}$ and

$$\mathcal{L}_{\theta_i}(\theta_j \lrcorner \mathrm{T}) \ = \ 0\,, \quad \theta_i \lrcorner \theta_j \lrcorner \mathrm{T} \ = \ 0 \ .$$

We will investigate the special case , where two of these 2-forms vanish, later.

Remark 4.2. We do not have any results in case of $|\mathrm{T}(\theta_1, \theta_2, \theta_3)| = \mu/2$.

4.2. **Special ∇-parallel vector fields.** There are special ∇-parallel vector fields (1-forms), namely

$$\mathcal{SP}^\nabla \ := \ \big\{\theta \ : \ \nabla^g \theta \ = \ 0 \text{ and } \theta \lrcorner \mathrm{T} \ = \ 0\big\} \subset \mathcal{P}^\nabla.$$

A consequence of the formula in Theorem 4.1 is the following

Corollary 4.2. *If* $\mathrm{T} \not\equiv 0$ *and* $\mathcal{R}^\nabla \not\equiv 0$, *then* $\dim(\mathcal{SP}^\nabla) \leq 2$.

Proposition 4.2. *If* $\theta \in \mathcal{SP}^\nabla$ *is special* ∇*-parallel, then*

$$\nabla^g_\theta \omega^3 \ = \ 0, \quad d(\theta \lrcorner \omega^3) \ = \ \theta \lrcorner d\omega^3, \quad \mathcal{L}_\theta(\theta \lrcorner \omega^3) \ = \ 0.$$

Proof. Since $\theta \lrcorner \mathrm{T} = 0$ we get

$$\nabla_\theta S \ = \ \nabla^g_\theta S \ + \ \frac{1}{2}\,\rho_*(\theta \lrcorner \mathrm{T})(S) \ = \ \nabla^g_\theta S$$

for any tensor S. Here ρ_* denotes action of $\mathfrak{so}(7)$ in the corresponding tensor representation. In particular,

$$\nabla^g_\theta \omega^3 \ = \ 0.$$

Since θ is ∇^g-parallel, we have $\nabla^g(\theta \lrcorner \omega^3) = \theta \lrcorner \nabla^g \omega^3$. Using an orthonormal frame with $\theta = e_7$ we compute the differential

$$\begin{aligned} d(\theta \lrcorner \omega^3) &= \sum_{i=1}^{7} \nabla^g_{e_i}(\theta \lrcorner \omega^3) \wedge e_i \;=\; \sum_{i=1}^{6} (\theta \lrcorner \nabla^g_{e_i}\omega^3) \wedge e_i \;+\; 0 \;=\; \sum_{i=1}^{6} \theta \lrcorner (\nabla^g_{e_i}\omega^3 \wedge e_i) \\ &= \sum_{i=1}^{6} \theta \lrcorner (\nabla^g_{e_i}\omega^3 \wedge e_i) \;+\; \theta \lrcorner (\nabla^g_{\theta}\omega^3 \wedge \theta) \;=\; \theta \lrcorner d\omega^3. \end{aligned}$$

Finally, $\mathcal{L}_\theta(\theta \lrcorner \omega^3) = \theta \lrcorner d(\theta \lrcorner \omega^3) = \theta \lrcorner \theta \lrcorner\ d\omega^3 = 0.$ □

Theorem 4.3. *Let (M^7, g, ω^3) be a compact, cocalibrated* G_2*-manifold and ∇ its characteristic connection. Suppose that* $\mathrm{Ric}^\nabla = 0$, $\|\mathrm{T}\|^2 = \mu^2 > 0$ *and* $\mathcal{R}^\nabla \not\equiv 0$. *Then the space of harmonic* 1*- forms coincides with* $\mathcal{SP}^\nabla$,

$$H^1(M^7;\mathbb{R}) \;=\; \big\{\theta \,:\, \Delta^g\theta \;=\; 0\big\} \;=\; \mathcal{SP}^\nabla \ .$$

In particular, the second Betti number is bounded, $b_2(M^7) \leq 2$.

Proof. The result follows directly from the Weitzenboeck formula for 1-forms and the link between Ric^g and the torsion form T,

$$0 \;=\; \int_{M^7} g(\Delta^g\theta\,,\,\theta) \;=\; \int_{M^7} ||\nabla^g\theta||^2 + \int_{M^7} \mathrm{Ric}^g(\theta,\theta) \;=\; \int_{M^7} ||\nabla^g\theta||^2 + \frac{1}{2}\int_{M^7} \|\theta \lrcorner \mathrm{T}\|^2.$$

□

4.3. **The case of two special ∇-parallel vector fields.** Suppose that there exist two special ∇-parallel vector fields θ_1, θ_2,

$$\nabla^g\,\theta_1 \;=\; \nabla^g\theta_2 \;=\; 0\;, \quad \theta_1 \lrcorner \mathrm{T} \;=\; \theta_2 \lrcorner \mathrm{T} \;=\; 0\;.$$

Then $\omega^3(\theta_1, \theta_2, -) = \theta_3$ is the third ∇-parallel (non-special) vector field and we have

$$\mathrm{T}(\theta_1,\theta_2,\theta_3) \;=\; 0, \quad [\theta_1,\theta_2] \;=\; [\theta_1,\theta_3] \;=\; [\theta_2,\theta_3] \;=\; 0.$$

The conditions $\theta_1 \lrcorner \mathrm{T} = \theta_2 \lrcorner \mathrm{T} = 0$ restrict the algebraic type of the torsion form. In fact, Theorem 4.1 yields that the possible torsion forms depend on two parameters only. Indeed, there are two possibilities. The first case:

$$a \;=\; \frac{2}{7}\,\mu,\; b \;=\; \frac{5}{7}\,\mu,\; m_1 \;=\; -\,\frac{8}{7}\,\mu,\; m_2 \;=\; m_3 \;=\; \frac{6}{7}\,\mu.$$

The second case:

$$a \;=\; \frac{2}{7}\,\mu,\; b \;=\; -\,\frac{2}{7}\,\mu,\; m_1 \;=\; \frac{6}{7}\,\mu,\; m_2 \;=\; \frac{6}{7}\,\mu,\; m_3 \;=\; -\,\frac{8}{7}\,\mu.$$

Introducing a new notation for the frame

$$f_1 \;:=\; e_3,\; f_2 \;:=\; e_4,\; f_3 \;:=\; e_5,\; f_4 \;:=\; e_6,\; f_5 \;:=\; e_7$$

we obtain the following formula for the torsion form:

$$\begin{aligned} \mathrm{T} &= (t_{125} + \mu/7)f_{125} + t_{245}(f_{135} + f_{245}) + t_{235}(-f_{145} + f_{235}) + (t_{345} + \mu/7)f_{345}\ , \\ b &= t_{125} \,+\, t_{345} \;=\; \frac{5}{7}\,\mu \ \text{ or } \ -\,\frac{2}{7}\,\mu \\ \mu^2 &= \|\mathrm{T}\|^2 \;=\; (t_{125} \,+\, \frac{\mu}{7})^2 \,+\, (t_{345} \,+\, \frac{\mu}{7})^2 \,+\, 2\,t_{245}^2 \,+\, 2\,t_{235}^2. \end{aligned}$$

If M^7 is complete, its universal covering splits into $N^5 \times \mathbb{R}^2$ and the torsion T as well as the form $\theta_3 = e_7 = f_5$ are forms on N^5. This follows form $\mathcal{L}_{\theta_i}\mathrm{T} = 0\,,\, \mathcal{L}_{\theta_i}\theta_3 = 0$ for

$i = 1, 2$. We reduced the dimension. $(N^5, g, \nabla, \mathrm{T}, \theta_3)$ is a 5-dimensional Riemannian manifold equipped with a torsion form T as well as a metric connection ∇ such that

$$d * \mathrm{T} \;=\; 0\;, d\,\mathrm{T} \;=\; 0,\; ||\mathrm{T}||^2 \;=\; 0,\; \mathrm{Ric}^{\nabla} \;=\; 0,\; \mathcal{R}^{\nabla} \;\not\equiv\; 0,\; \mathfrak{hol}(\nabla) \subset \;\mathfrak{su}(2)\; \subset\; \mathfrak{g}_2$$

hold. θ_3 is ∇-parallel on N^5,

$$\nabla\theta_3 \;=\; 0,\; d\theta_3 \;=\; \theta_3 \lrcorner \mathrm{T},\; \mathrm{T} \;=\; \theta_3 \wedge d\,\theta_3,\; 0 \;=\; d\,\mathrm{T} \;=\; d\theta_3 \wedge d\theta_3 .$$

Consider the case of $b = -2\mu/7$. Then

$$t_{125} + \frac{\mu}{7} \;=\; -t_{345} \;-\; \frac{\mu}{7}$$

and we obtain

$$*\,\mathrm{T} = \;-\,\theta_3 \lrcorner \mathrm{T} \;=\; -d\theta_3, \quad *d\theta_3 \;=\; -\mathrm{T} \;=\; -\,d\theta_3 \wedge \theta_3 .$$

We multiply the latter equation by $d\theta_3$:

$$\|d\theta_3\|^2 \;=\; d\theta_3 \wedge *d\theta_3 \;=\; -\theta_3 \wedge d\theta_3 \wedge d\theta_3 \;=\; 0.$$

Consequently, $b = -2\,\mu/7$ implies that the torsion form vanishes, $\mathrm{T} = 0$, i.e. the second case is impossible.

We observe that there are three ∇-parallel 2-forms on N^5, namely,

$$\Omega^2_i \;:=\; \theta_i \lrcorner \big(\omega^3 - \theta_1 \wedge \theta_2 \,\wedge\, \theta_3\big).$$

Consequently, $\mathfrak{hol}(\nabla) \subset\; \mathfrak{su}(2)$. We can express these forms in our local frame,

$$\Omega^2_1 \;=\; f_{13} \,-\, f_{24}, \quad \Omega^2_2 \;=\; -\,f_{14} \,-\, f_{23}, \quad \Omega^2_3 \;=\; f_{12} \,+\, f_{34}.$$

Remark that

$$\big(\theta_3 \lrcorner \mathrm{T}\,,\, \Omega^2_1\big) \;=\; \big(\theta_3 \lrcorner \mathrm{T}\,,\, \Omega^2_2\big) \;=\; 0\;, \quad \big(\theta_3 \lrcorner \mathrm{T}\,,\, \Omega^2_3\big) \;=\; b \,+\, \frac{2}{7}\,\mu \;=\; \mu$$

holds.

Theorem 4.4. *The kernel of* T

$$E^2 \;:=\; \big\{X \in TN^5 \;:\; X \lrcorner \mathrm{T} \;=\; 0\big\}$$

is a 2*-dimensional subbundle of* TN^5*. The tangent bundle splits into two subbundles of dimension* 2 *and* 3*, respectively,*

$$TN^5 \;=\; E^2 \;\oplus\; (E^2)^{\perp}.$$

θ_3 *belongs to* $(E^2)^{\perp}$ *and the torsion form is given by*

$$\mathrm{T} \;=\; \mu\, f^*_1 \wedge f^*_2 \wedge \theta_3,$$

where f^*_1, f^*_2, θ_3 *is an orthonormal basis in* $(E^2)^{\perp}$*. Both subbundles are involutive and* N^5 *splits locally (but the* 2*- und* 3*-dimensional leaves are not totally geodesic).*

Proof. We compute the determinant of the skew symmetric endomorphism $\theta_3 \lrcorner \mathrm{T}$ on the space of all vectors being orthogonal to θ_3,

$$\mathrm{Det}(\theta_3 \lrcorner \mathrm{T}) \;=\; \frac{1}{4}\big(-\,b^2 \,-\, \frac{4}{7}\,b\,\mu \,+\, \frac{45}{49}\,\mu^2\big)^2 \;=\; 0\;.$$

This proves that the dimension of E^2 equals two. Let $f^*_1, f^*_2, f^*_3, f^*_4, f^*_5 \;=\; \theta_3$ be an orthonormal frame such that

$$\mathrm{Lin}(f^*_1\,,\, f^*_2\,,\, f^*_5) \;=\; \big(E^2\big)^{\perp}\;, \quad \mathrm{Lin}(f^*_3\,,\, f^*_4) \;=\; E^2\;.$$

Since μ is constant and $d\,\mathrm{T} = d * \mathrm{T} = 0$ we have

$$d\big(f_1^* \wedge f_2^* \wedge f_5^*\big) \ = \ 0\ , \quad d\big(f_3^* \wedge f_4^*\big) \ = \ 0\ .$$

We differentiate the equations $f_3^* \wedge f_3^* \wedge f_4^* = 0\,,\ f_4^* \wedge f_3^* \wedge f_4^* = 0$,

$$\begin{aligned} 0 \ &= \ df_3^* \wedge (f_3^* \wedge f_4^*) \ - \ f_3^* \wedge d(f_3^* \wedge f_4^*) \ = \ df_3^* \wedge (f_3^* \wedge f_4^*) \\ 0 \ &= \ df_4^* \wedge (f_3^* \wedge f_4^*) \ - \ f_4^* \wedge d(f_3^* \wedge f_4^*) \ = \ df_4^* \wedge (f_3^* \wedge f_4^*)\ . \end{aligned}$$

By the Frobenius Theorem, the bundle $(E^2)^{\perp}$ is involutive. Similarly we have

$$df_1^* \wedge (f_1^* \wedge f_2^* \wedge f_5^*) \ = \ df_2^* \wedge (f_1^* \wedge f_2^* \wedge f_5^*) \ = \ df_5^* \wedge (f_1^* \wedge f_2^* \wedge f_5^*) \ = \ 0$$

and the bundle E^2 is involutive. □

This splitting is not ∇-parallel ($\nabla \mathrm{T} \neq 0$), but the flow of θ_3 preserves the splitting ($\mathcal{L}_{\theta_3}\mathrm{T} = 0$). The Ricci tensor preserves the splitting, too. Indeed, it depends only on T and we compute easily:

Theorem 4.5. *The Ricci tensor* Ric^g *preserves the splitting of the tangent bundle and*

$$\mathrm{Ric}^g_{|E^2} \ = \ 0, \quad \mathrm{Ric}^g_{|(E^2)^{\perp}} \ = \ \frac{1}{2}\mu^2\,\mathrm{Id}.$$

In particular, the Ricci tensor of (N^5, g) *has constant eigenvalues, and these are* 0 *and* $\mu^2/2 > 0$.

The 2-form $d\theta_3$ is invariant under the flow of θ_3,

$$\mathcal{L}_{\theta_3}\big(d\theta_3\big) \ = \ 0 \quad \text{and} \quad d\theta_3 \wedge d\theta_3 \ = \ 0.$$

If the orbit space $Z^4 := N^5/\theta_3$ is smooth, its tangent bundle splits into two involutive 2-dimensional subbundles. $d\theta_3$ defines a 2-form on Z^4 satisfying all the conditions of Theorem 2.1. However, we have an additional condition for $(N^5, g, \nabla, \mathrm{T}, \theta_3)$, namely the holonomy of ∇ should be contained in $\mathfrak{su}(2) \subset \mathfrak{g}_2$ and the holonomy representation is in $\mathbb{C}^2 \subset \mathbb{R}^5$. This is equivalent to the condition that there are three ∇-parallel 2-forms $\Omega^2_1, \Omega^2_2, \Omega^2_3$. The 2-form Ω^2_3 plays a special role on N^5. Indeed , it projects down to a Kähler form on Z^4.

Proposition 4.3.

$$\nabla\,\Omega^2_3 \ = \ 0, \quad d\,\Omega^2_3 \ = \ 0, \quad \mathcal{L}_{\theta_3}\Omega^2_3 \ = \ 0.$$

In particular, if Z^4 *is smooth, then* $\Omega^2_3 \in \Lambda^2_+(Z^4)$ *defines a* ∇^g*-parallel, self-dual* 2*-form on* Z^4.

Proof. Using the frame $f_1, \ldots, f_5$ one easily computes the formula

$$\Omega^2_3 \ = \ \frac{1}{\mu}\big(\, * \mathrm{T} \, + \, d\,\theta_3\big) \ = \ \frac{1}{\mu}\big(\, * \mathrm{T} \, + \, \theta_3 \lrcorner \mathrm{T}\big)\ .$$

Since $d * \mathrm{T} = 0$ we obtain $d\,\Omega^2_3 = 0$. Moreover, $\mathcal{L}_{\theta_3}\mathrm{T} = 0$, and

$$\mathcal{L}_{\theta_3}\Omega^2_3 \ = \ \frac{1}{\mu}\,\mathcal{L}_{\theta_3}(d\theta_3) \ = \ \frac{1}{\mu}\big(\theta_3 \lrcorner (\theta_3 \lrcorner \mathrm{T})\big) \ = \ 0\ .$$ □

A similar algebraic computation yields the following formulas.

Proposition 4.4.

$$\begin{aligned} d\,\Omega^2_1 \ &= \ \mu\,\Omega^2_2 \wedge \theta_3\ , \quad & d\,\Omega^2_2 \ &= -\,\mu\,\Omega^2_1 \wedge \theta_3\ , \\ \mathcal{L}_{\theta_3}\Omega^2_1 \ &= \ \mu\,\Omega^2_2\ , & \mathcal{L}_{\theta_3}\Omega^2_2 \ &= \ -\,\mu\,\Omega^2_1\ . \end{aligned}$$

Proof. Since the 2-forms are ∇-parallel, we can compute the derivatives using the formula (see [2])

$$d\,\Omega^2 \;=\; \sum_{j=1}^{5}(f_j \lrcorner\, \Omega^2) \wedge (f_j \lrcorner\, \mathrm{T})\ . \qquad \square$$

Remark 4.3. In the frame $f_1^*, \ldots, f_5^*$ we have $\Omega_3^2 = f_1^* \wedge f_2^* + f_3^* \wedge f_4^*$, too. In particular, Ω_3^2 is completely defined by T and θ_3. If Z^4 is smooth and compact, then $Z^4 = S^2 \times T^2$, see [6], and the connection ∇ on $M^7 = N^5 \times \mathbb{R}^2 = S^3 \times T^2 \times \mathbb{R}^2$ becomes flat.

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Thomas Friedrich
Institut für Mathematik
Humboldt-Universität zu Berlin
Sitz: WBC Adlershof
D-10099 Berlin, Germany
friedric@mathematik.hu-berlin.de