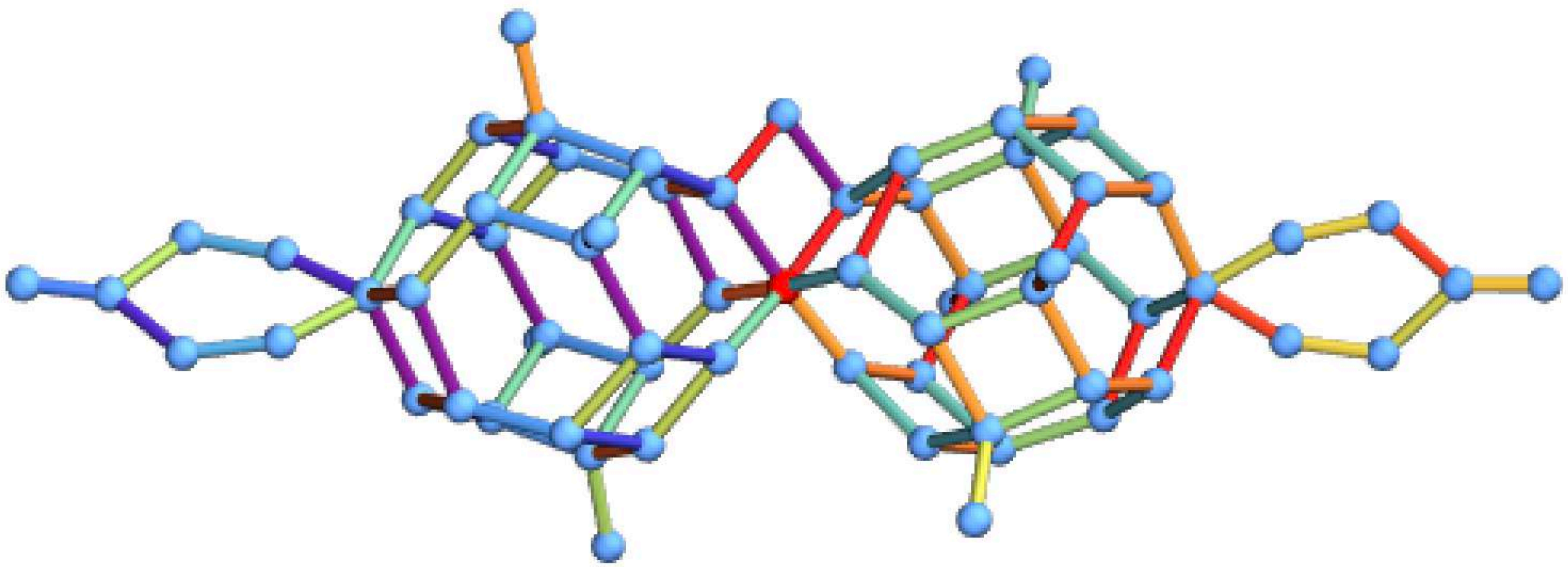
Permutohedra for knots and quivers



Piotr Sułkowski

Faculty of Physics, University of Warsaw

Algebra, Geometry & Physics seminar – July, 2021

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Jakub Jankowski[1,3] , **Piotr Kucharski**[2,3] , **Hélder Larraguível**[3] , **Dmitry Noshchenko**[3] , **and Piotr Sułkowski**[2,3]

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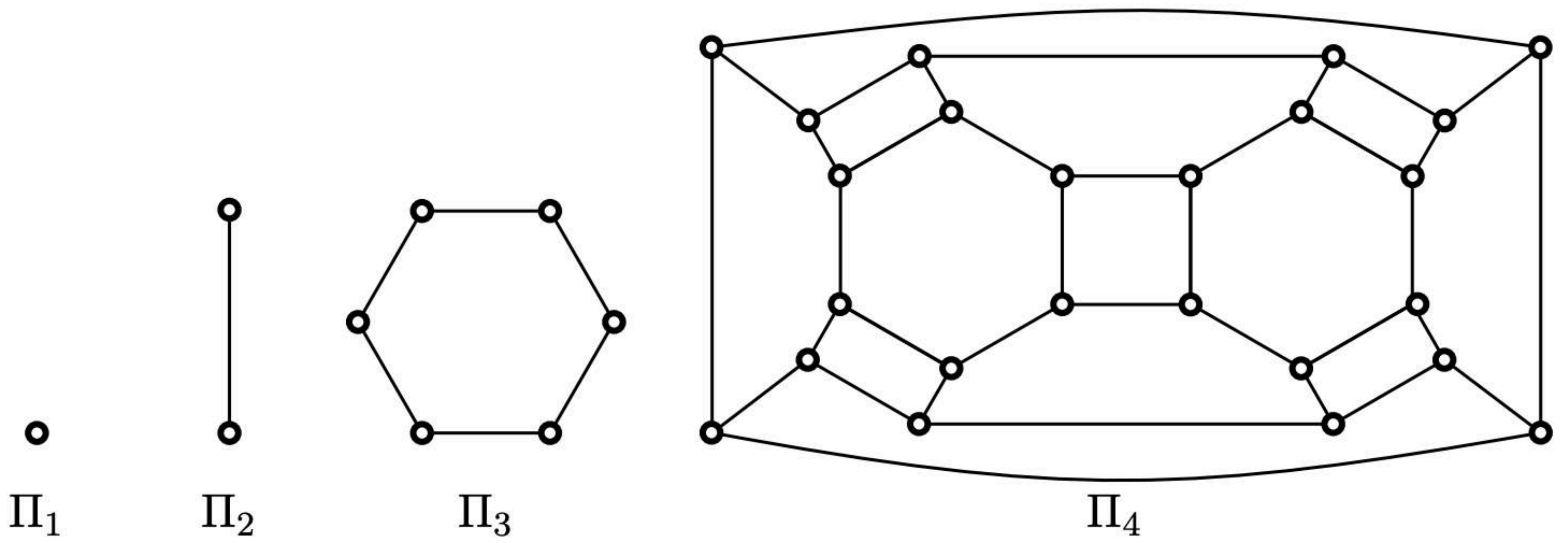
Abstract: The knots-quivers correspondence states that various characteristics of a knot are encoded in the corresponding quiver and the moduli space of its representations. However, this correspondence is not a bijection: more than one quiver may be assigned to a given knot and encode the same information. In this work we study this phenomenon systematically and show that it is generic rather than exceptional. First, we find conditions that characterize equivalent quivers. Then we show that equivalent quivers arise in families that have the structure of permutohedra, and the set of all equivalent quivers for a given knot is parameterized by vertices of a graph made of several permutohedra glued together. These graphs can be also interpreted as webs of dual 3d $\mathcal{N} = 2$ theories. All these results are intimately related to properties of homological diagrams for knots, as well as to multi-cover skein relations that arise in counting of holomorphic curves with boundaries on Lagrangian branes in Calabi-Yau three-folds.

Permutohedron

Permutohedron: (n-1)-dimensional polytope whose vertices represent permutations of n objects, and edges correspond to transpositions of adjacent neighbours.



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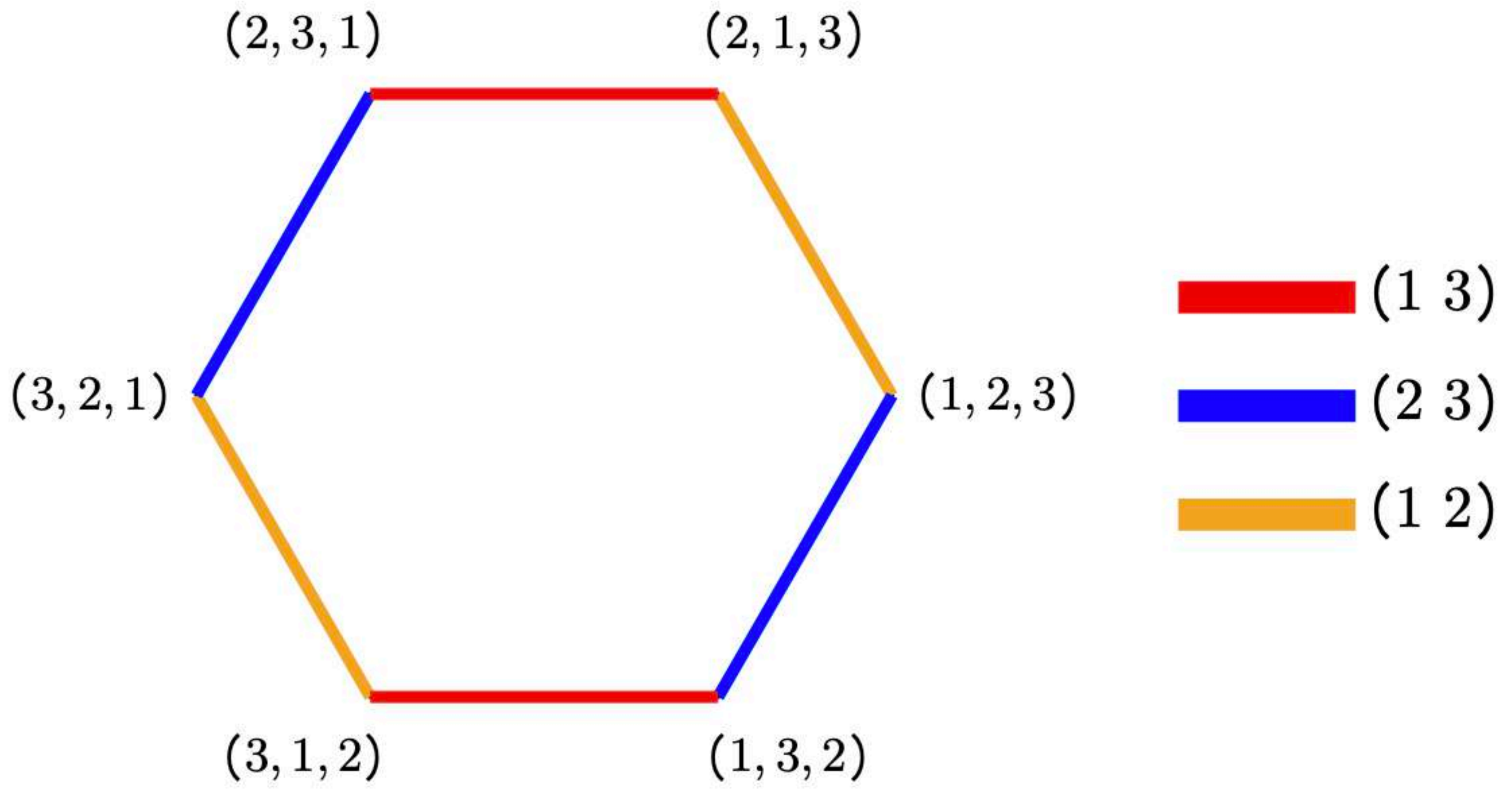

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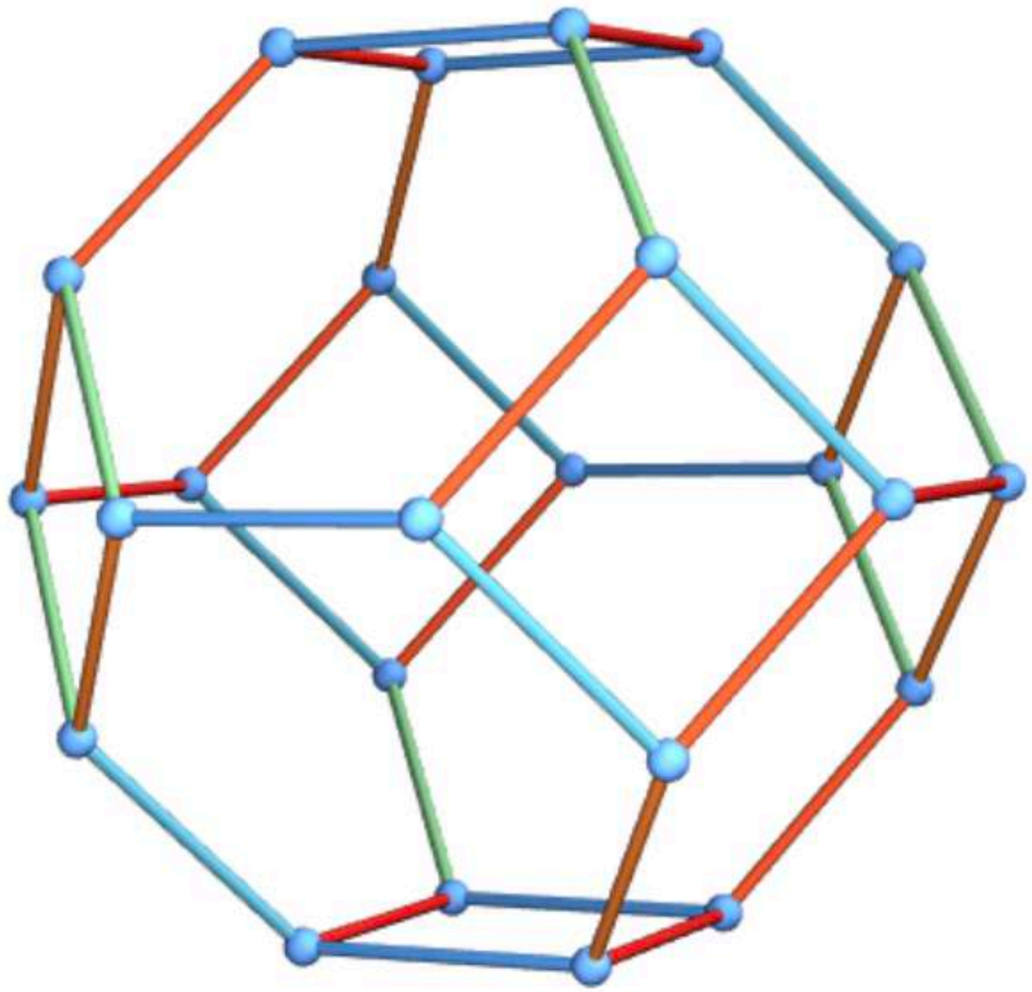
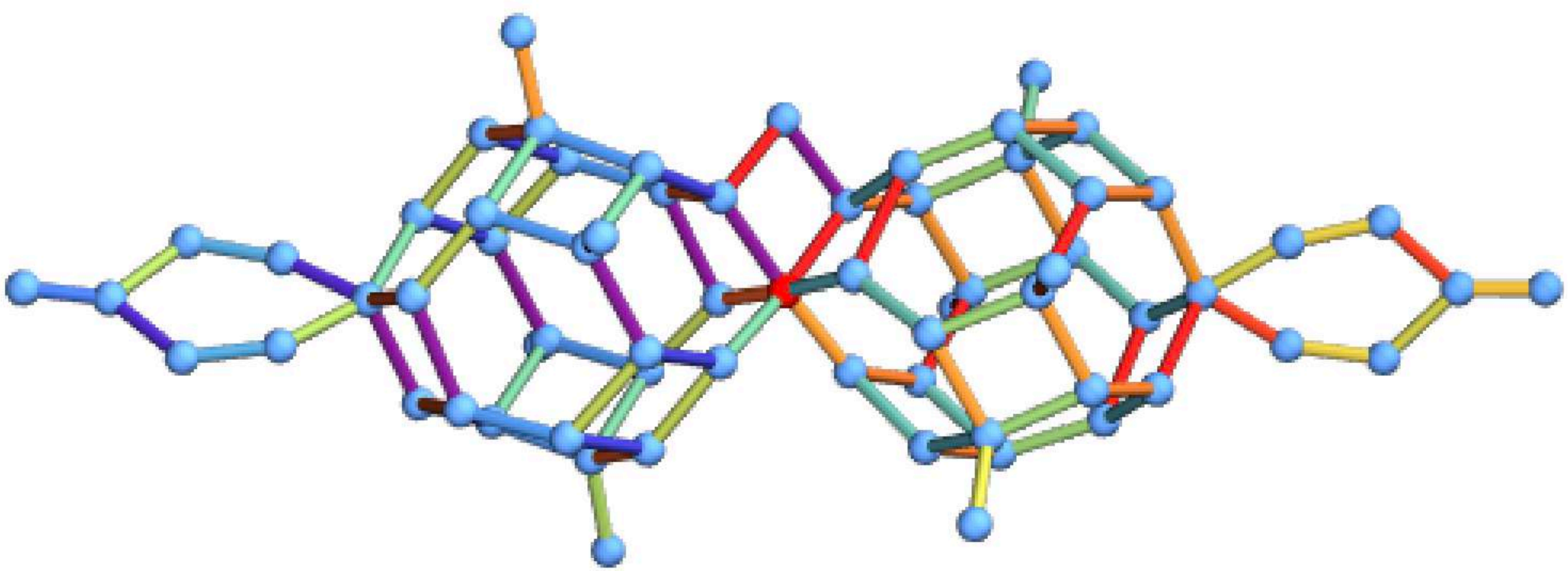


Permutohedra for knots and quivers

Permutohedra graphs – graphs made of several permutohedra, whose vertices represent equivalent **quivers** associated to **knots**.



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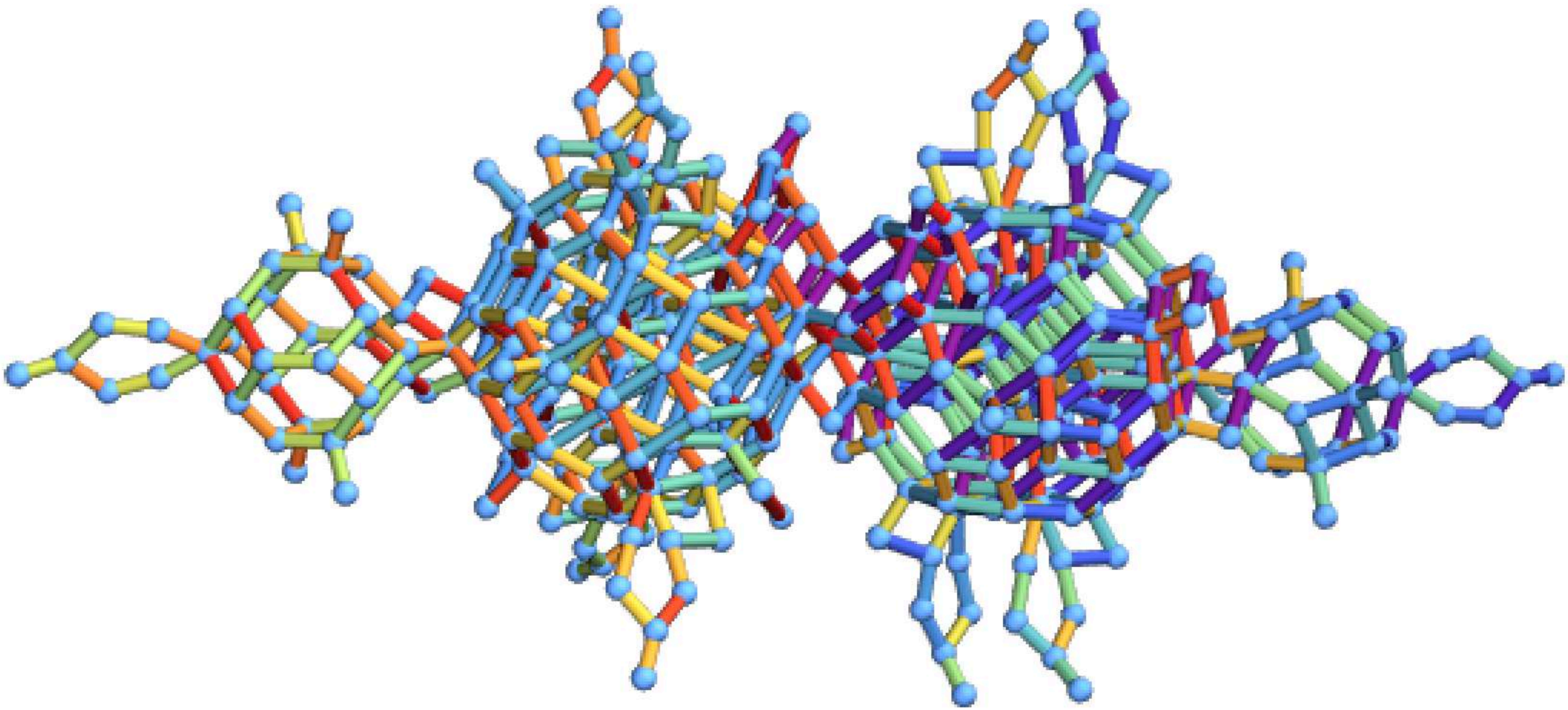
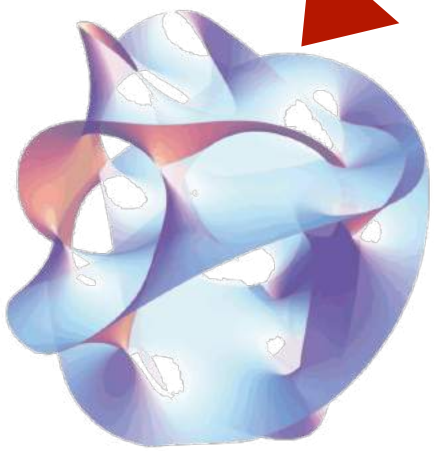


Compactification and dualities

Superstring theory – effective theory in 4-dim follows from compactification of 10-dim string theory on a Calabi-Yau manifold.

10-dim = R^4 x 

(Beyond) Standard Model theory

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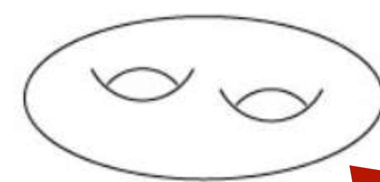
6-dim = $\mathbf{R^4}$ x 

N=2 SUSY gauge theory

2-dim CFT amplitudes (Liouville, Toda)

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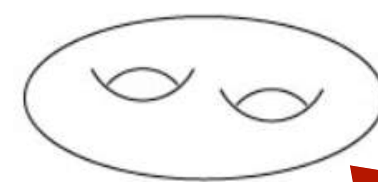
6-dim = R^4 x 

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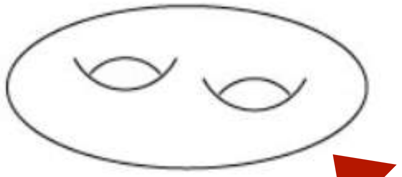
6-dim = R^4 x 

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$$S = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

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Polynomial knot invariants from Wilson loop observables:

$$P_R(a,q) = \left\langle \mathrm{Tr}_R \, e^{\oint A} \right\rangle = \int \mathcal{D}A \left(\mathrm{Tr}_R \, e^{\oint A} \right) e^{\frac{ik}{4\pi} S}$$

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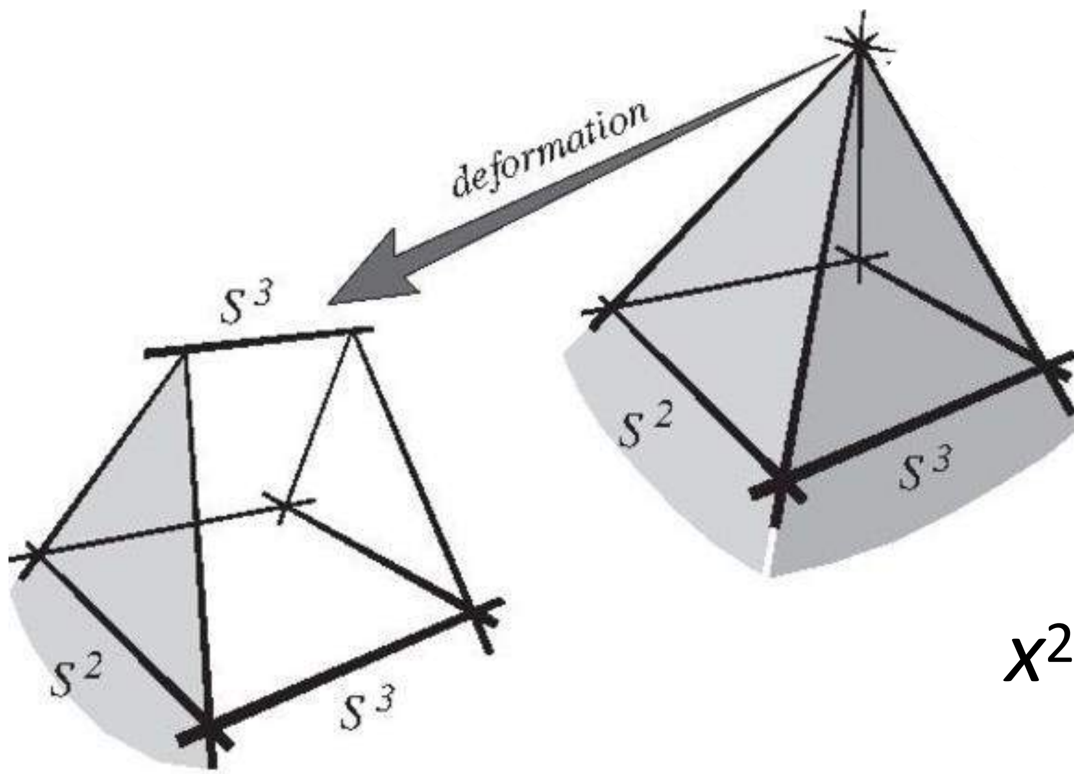
HOMFLY-PT polynomial for SU(N) gauge group:

$$q = e^{\frac{2\pi}{k+N}}, \qquad a = q^N$$

Jones polynomial for SU(2), Alexander polynomial for a=1

Topological strings and open-closed duality

Chern-Simons theory on **S**^{3} arises as an effective description of A-model open topological string theory in deformed conifold T^***S**^{3}, with appropriate boundary conditions (N branes) on **S**^{3} (*Witten, 1993*).



T^***S**^{3}

Singular conifold:

$$x^2 + y^2 + z^2 + w^2 = 0$$

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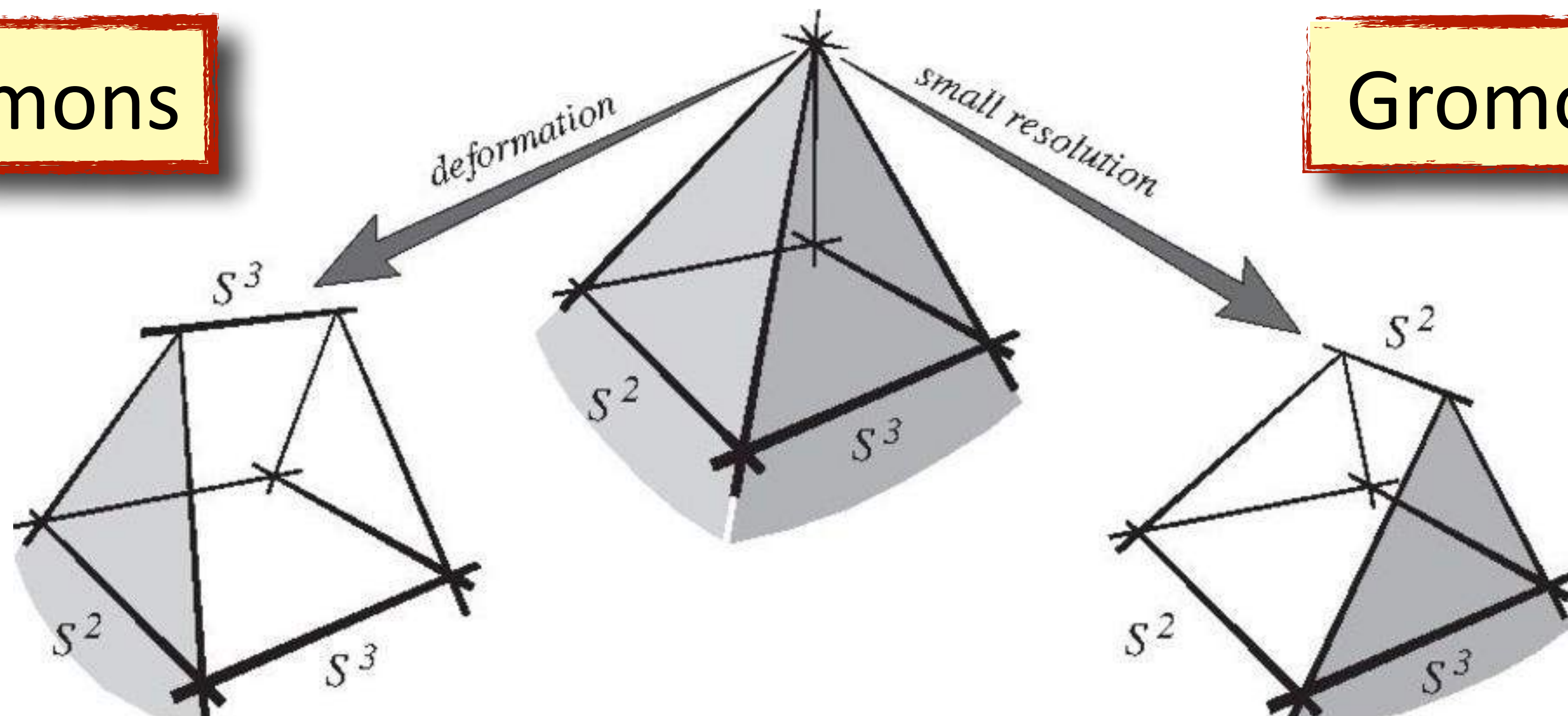

After a *geometric transition*, in the ’t Hooft limit, $T^*\mathbf{S}^3$ is replaced by the resolved conifold X (with non-trivial **S**^{2}), N branes disappear, and we are left with A-model **closed** topological string theory (*Gopakumar-Vafa, 1998*).

$$Z^{\text{closed}} = \exp\Big(\sum_{g=0}^{\infty} g_s^{2g-2} F_g(Q)\Big)$$

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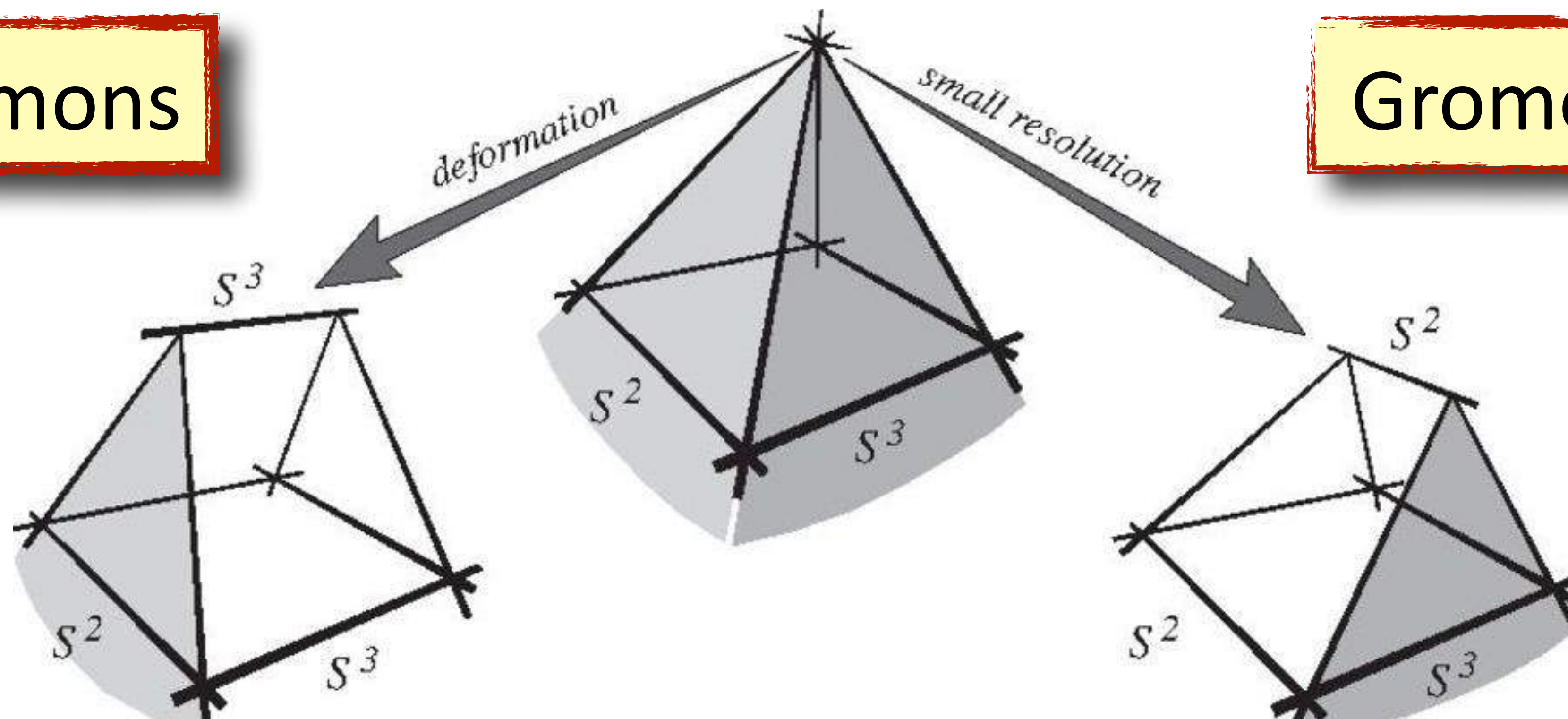

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Knots arise once we introduce an extra lagrangian brane, which intersects $\mathbf{S}^3$, along a knot K. This brane survives the geometric transition.

M-theory, knots and BPS states

Embed the above system in M-theory. Chern-Simons theory on $\mathbf{S}^3$ engineered by N M5-branes in deformed conifold $T^*\mathbf{S}^3$. A knot K engineered by extra M5-branes on lagrangian L_K. What is effective SUSY theory in 3 spacetime dimensions?

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\text{space-time}: & \mathbb{R} \times T^*\mathbf{S}^3 \times M_4 \\
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Knot invariants of K, computed by Chern-Simons theory on the initial $\mathbf{S}^3$, are encoded in (conjecturally) integral BPS invariants (*Labastida-Marino-Ooguri-Vafa, 2000*) in the effective SUSY theory on ($\mathbf{R}$ x D).

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Ooguri-Vafa (LMOV) invariants

Brane amplitude as generating function of colored polynomials:

$$\sum_R P_R(a,q)\mathrm{Tr}_R V = \exp\Big(\sum_{n=1}^{\infty}\sum_R \frac{1}{n} f_R(a^n,q^n)\mathrm{Tr}_R V^n\Big)$$

with f_R enumerating bound states of D2-D4 branes:

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Knots-quivers correspondence

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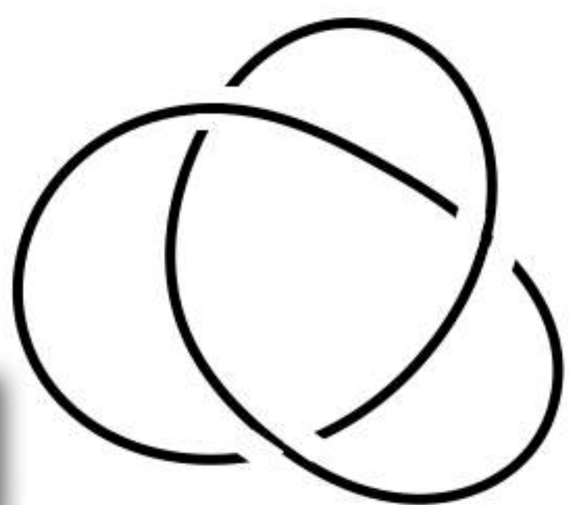
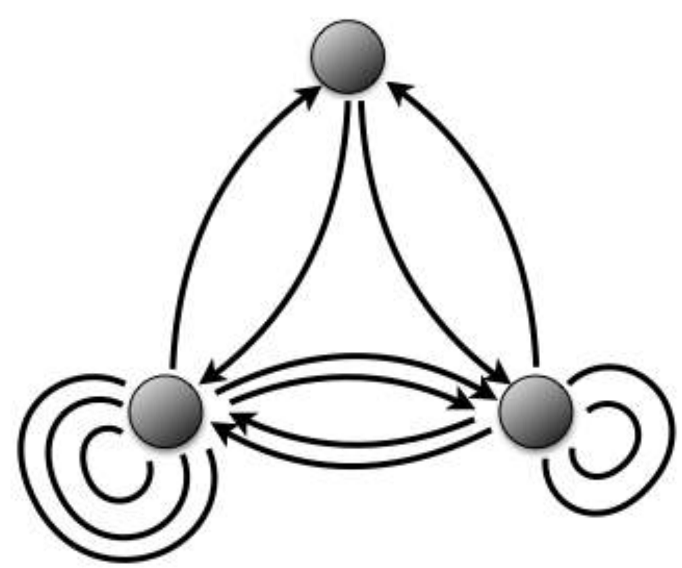
Kucharski-Reineke-Stosic-PS (arXiv: 1707.02991, 1707.04017)

BPS states enumerated by LMOV invariants are bound states of certain "elementary" states, whose interactions are encoded in a quiver diagram. Nodes of a quiver correspond to those "elementary" states, and arrows to interactions.

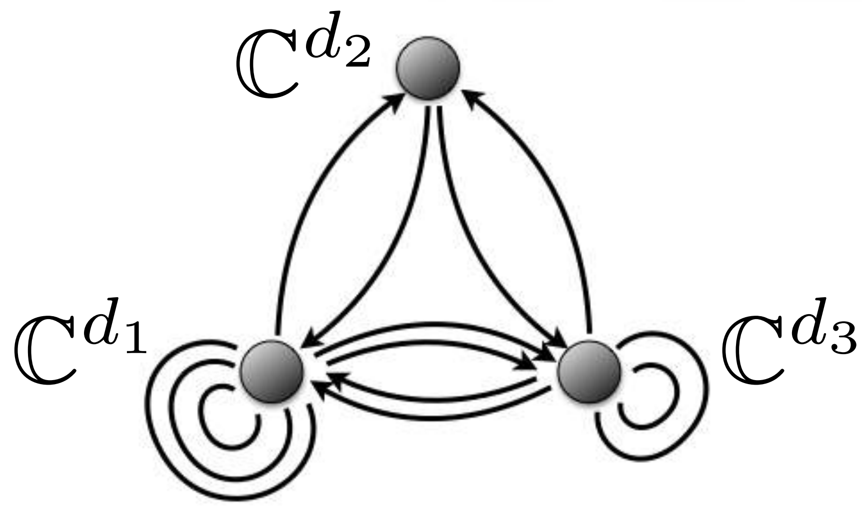
Calabi-Yau description





Spacetime description

Quiver representation theory



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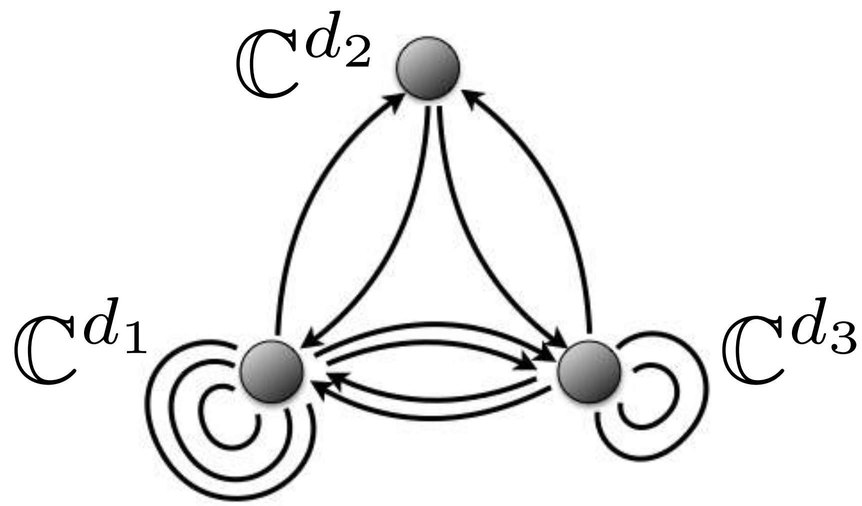

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$$= \prod_{(d_1,\ldots,d_m)\neq 0} \prod_{j\in\mathbb{Z}} \prod_{k\geq 0} \left(1-(x_1^{d_1}\cdots x_m^{d_m})q^{j+2k+1}\right)^{(-1)^{j+1}\Omega_{d_1,\ldots,d_m;j}}$$

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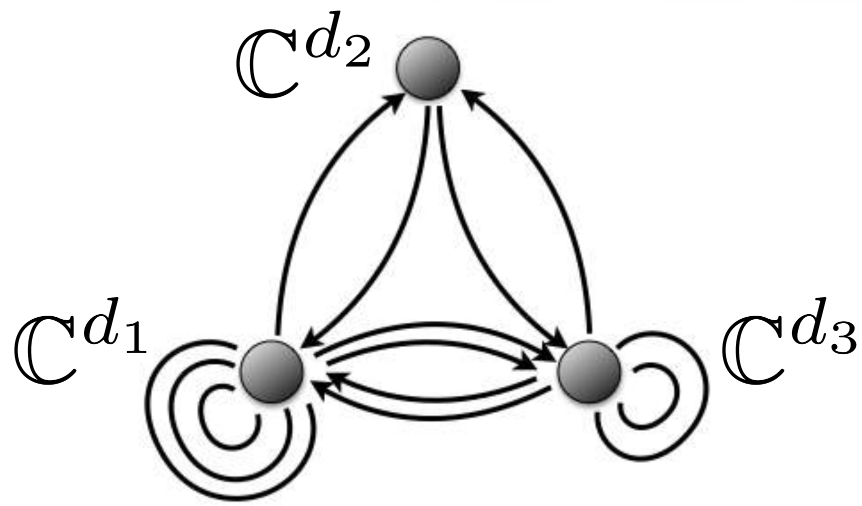

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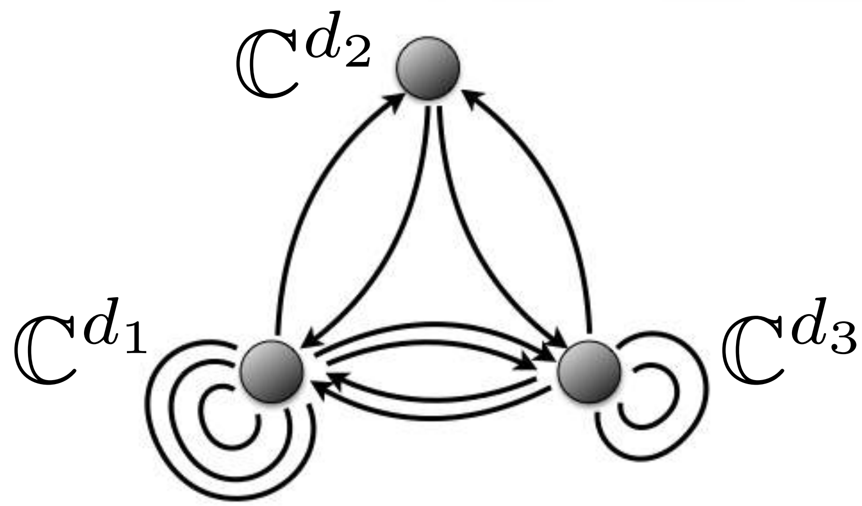

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With appropriate identification of variables, generating function of colored HOMFLY-PT polynomials can be written in the form of motivic generating function, for some particular symmetric matrix *C*

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$$x_i = x a^{a_i} q^{l_i - 1} (-1)^{t_i}$$

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Knots	Quivers
Homological degrees, framing	Number of loops

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$$P(x) = \sum_{r=0}^{\infty} \overline{P}_r(a,q) x^r = \sum_{d_1,\ldots,d_m \geq 0} q^{\sum_{i,j} C_{i,j} d_i d_j} x^{d_1+\ldots+d_m} \frac{\prod_{i=1}^m q^{l_i d_i} a^{a_i d_i} (-1)^{t_i d_i}}{\prod_{i=1}^m (q^2;q^2)_{d_i}}$$

Note: infinite number of colored polynomials / LMOV invariants encoded in a finite number of parameters of a matrix *C*.

Knots	Quivers
Homological degrees, framing	Number of loops
Colored HOMFLY-PT	Motivic generating series

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LMOV invariants	Motivic DT-invariants $\in \mathbb{N}$

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Algebra of BPS states	Cohom. Hall Algebra

Quivers and HOMFLY-PT homology

Recall – colored HOMFLY-PT polynomials arise as Euler characteristics of coloured HOMFLY-PT homologies:

$$P_r(a,q) = P_r(a,q,-1) = \sum_{i,j,k} a^i q^j (-1)^k \dim \mathcal{H}^{S^r}_{ijk}(K).$$

Some homological information is encoded in superpolynomials:

$$P_r(a,q,t) = \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}^{S^r}_{ijk}(K) \equiv \sum_{i \in \mathcal{G}_r(K)} a^{a_i^{(r)}} q^{q_i^{(r)}} t^{t_i^{(r)}}$$

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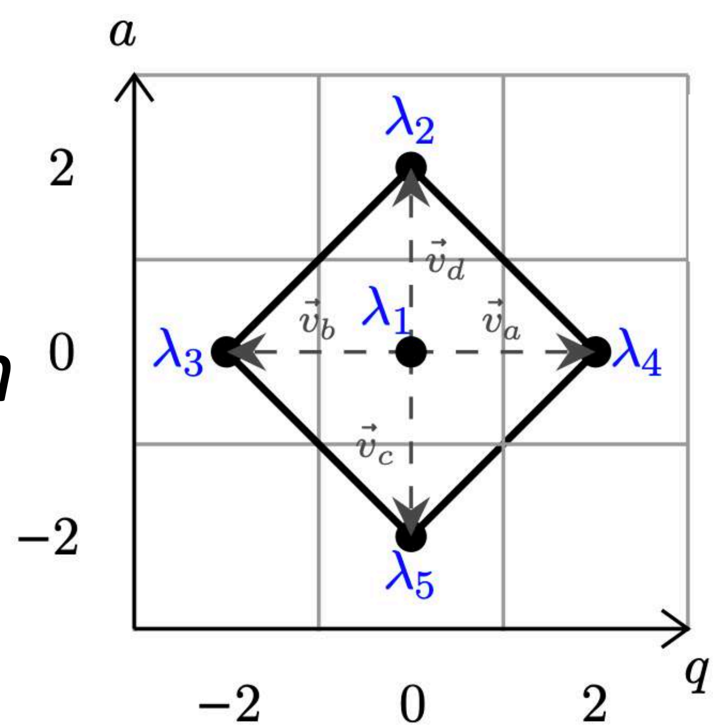
Homological diagram for figure-8 knot



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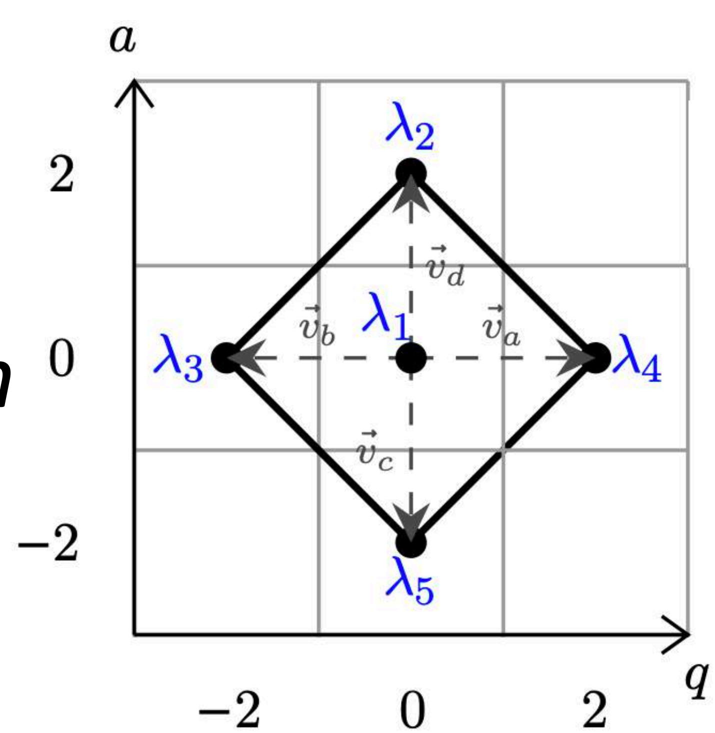

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Relation to quivers:

$$t_i^{(1)} \equiv t_i = C_{i,i}$$

Examples

Colored polynomial for trefoil: $P_r(a,q) = \frac{a^{2r}}{q^{2r}} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} q^{2k(r+1)} \prod_{i=1}^{k} (1 - a^2 q^{2(i-2)})$

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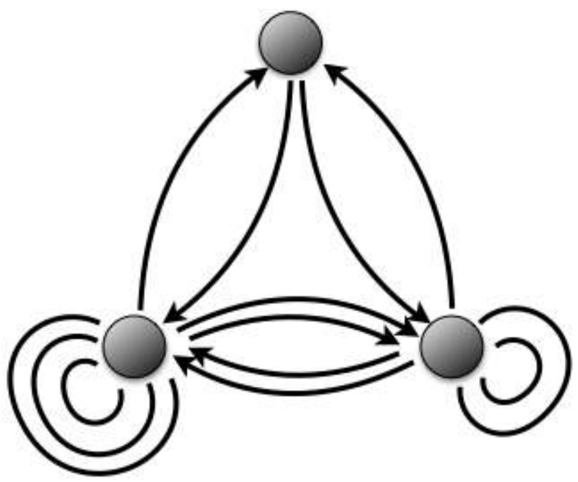
Quiver form follows from: $\begin{bmatrix} r \\ k \end{bmatrix} (\frac{a^2}{q^2}; q^2)_k = \sum_{i=0}^{k} \frac{(q^2;q^2)_r \left(-\frac{a^2}{q^2}\right)^i q^{i(i-1)}}{(q^2;q^2)_{r-k}(q^2;q^2)_i(q^2;q^2)_{k-i}}.$

We find: $C^{T_{2,3}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$



Examples – torus knots

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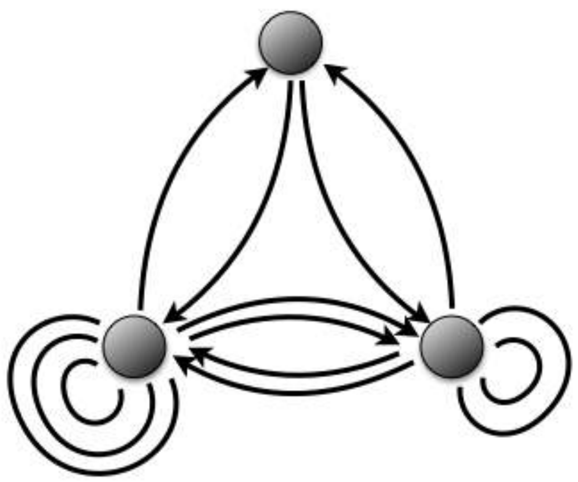


(2,5) torus knot: $C^{T_{2,5}} = \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 & 4 \\ 3 & 3 & 4 & 4 & 5 \end{bmatrix}$

Examples – torus knots

$$C^{T_{3,4}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 4 & 4 & 5 \\ 3 & 3 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix}$$

$$C^{T_{3,5}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 7 \\ 1 & 2 & 3 & 3 & 5 & 5 & 7 \\ 2 & 3 & 4 & 4 & 5 & 5 & 7 \\ 3 & 3 & 4 & 4 & 6 & 5 & 7 \\ 4 & 5 & 5 & 6 & 6 & 6 & 7 \\ 5 & 5 & 5 & 5 & 6 & 6 & 7 \\ 7 & 7 & 7 & 7 & 7 & 7 & 8 \end{bmatrix}$$

$$C^{T_{3,7}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 5 & 6 & 7 & 8 & 9 & 11 \\ 1 & 2 & 3 & 3 & 5 & 5 & 6 & 7 & 7 & 9 & 9 & 11 \\ 2 & 3 & 4 & 4 & 5 & 6 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 3 & 4 & 4 & 5 & 5 & 7 & 7 & 7 & 9 & 9 & 11 \\ 4 & 5 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 9 & 10 & 11 \\ 5 & 5 & 6 & 5 & 6 & 6 & 8 & 7 & 7 & 9 & 9 & 11 \\ 5 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 10 & 11 \\ 6 & 7 & 7 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 10 & 11 \\ 7 & 7 & 8 & 7 & 8 & 7 & 9 & 8 & 8 & 9 & 9 & 11 \\ 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 & 10 & 11 \\ 9 & 9 & 10 & 9 & 10 & 9 & 10 & 10 & 9 & 10 & 10 & 11 \\ 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 12 \end{bmatrix}.$$

Examples – 6_2 knot

$$C^{6_2} = \begin{bmatrix} -2 & -2 & -1 & -1 & -1 & -1 & 0 & -1 & 1 & 1 & 1 \\ -2 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \end{bmatrix}$$

Examples – 6_3 knot

$$C^{6_3} = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & -1 & -2 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & -3 & 0 & -1 & -2 & -3 & -1 & 0 & -2 & -2 \\ -1 & -2 & -2 & -3 & -3 & -1 & -1 & -2 & -3 & -1 & -1 & -2 & -2 \\ 0 & 1 & 1 & 0 & -1 & 2 & 1 & 0 & -1 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 2 & 1 & 1 & 0 \\ -1 & -1 & 0 & -2 & -2 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 \\ -1 & -2 & -2 & -3 & -3 & -1 & -1 & -2 & -2 & 0 & -1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -1 & 2 & 2 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 & 2 & 2 & 1 & 0 \\ -1 & 0 & 0 & -2 & -2 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & -2 & -2 & -1 & 0 & -2 & -2 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Quivers determined for:

- all knots up to 6 crossings
- $(2,2p+1)$ torus knots, for all p
- $(2,2p)$ torus links
- $(3,3p+1)$ and $(3,3p+2)$ torus knots
- an infinite family of twist knots
- an infinite family of rational and arborescent knots (*Stosic-Wedrich*)

How unique is the correspondence?

Knot		Equivalent quivers
Unknot	0_1	1
Torus knots $T_{2,2p+1}$	3_1	1
	5_1	3
	7_1	13
	9_1	68
	11_1	405
	13_1	2 684
	15_1	19 557
	$\vdots$	$\vdots$
	$(2p+1)_1$	$\sim 2p!$
Twists knots $TK_{2\|p\|+2}$	4_1	2
	6_1	141
	8_1	36 555
Twists knots TK_{2p+1}	5_2	12
	7_2	1 983
Stand-alone examples	6_2	3 534
	6_3	142 368
	7_3	109 636

Local equivalence

arXiv: 2105.11806

Theorem 6. *Consider a quiver Q corresponding to the knot K and another symmetric quiver Q' such that $Q'_0 = Q_0$ and $\lambda'_i = \lambda_i$ $\forall i \in Q_0$ (λ_i comes from the knots-quivers change of variables). If Q and Q' are related by a sequence of disjoint transpositions, each exchanging non-diagonal elements*

$$C_{ab} \leftrightarrow C_{cd}, \qquad C_{ba} \leftrightarrow C_{dc},$$

for some pairwise different $a, b, c, d, \in Q_0$, such that

$$\lambda_a \lambda_b = \lambda_c \lambda_d$$

and

$$C_{ab} = C_{cd} - 1, \qquad C_{ai} + C_{bi} = C_{ci} + C_{di} - \delta_{ci} - \delta_{di}, \quad \forall i \in Q_0,$$

or

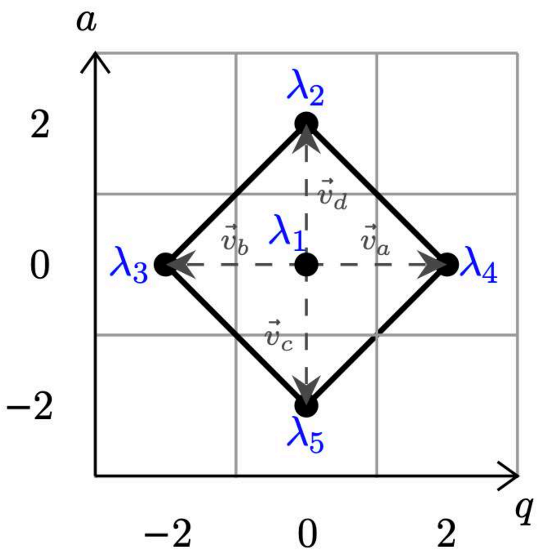
$$C_{cd} = C_{ab} - 1, \qquad C_{ci} + C_{di} = C_{ai} + C_{bi} - \delta_{ai} - \delta_{bi}, \quad \forall i \in Q_0,$$

then Q and Q' are equivalent in the sense of the definition 4.

Local equivalence

Proof: follows from comparison of quiver generating series *P*(*x*) for *Q* and *Q'*. Agreement at the order x^2 leads to the *center of mass* condition (i.e. the center of mass for nodes (*a,b*) coincides with the center of mass for nodes (*c,d*)).

$$\lambda_a \lambda_b = \lambda_c \lambda_d$$



Local equivalence

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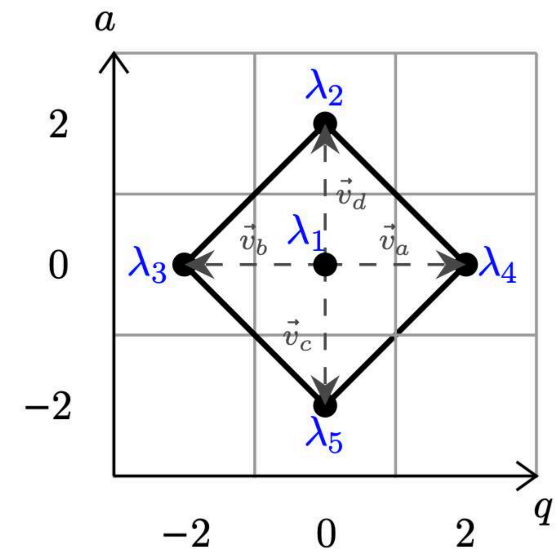

Agreement at the order x^3 leads to the conditions of the form:

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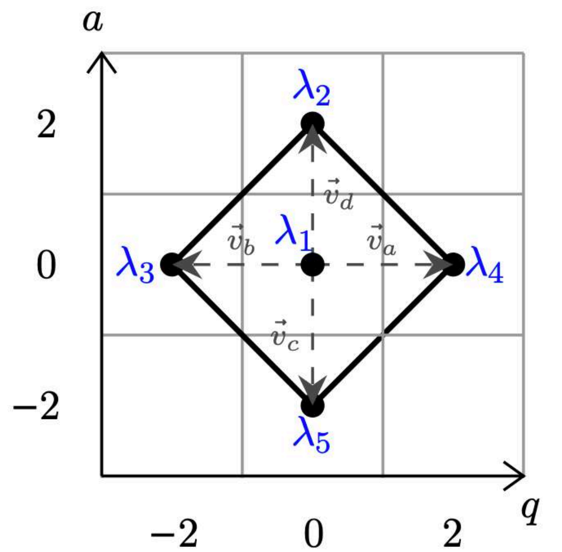

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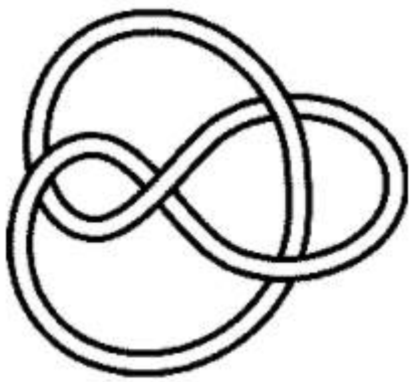
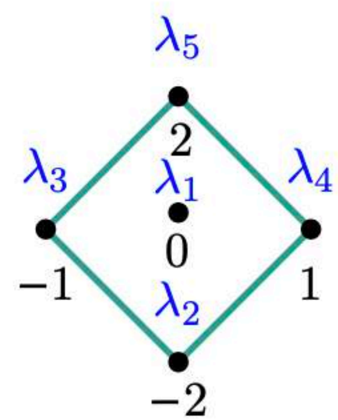
The agreement at the order x^3 asserts the agreement to all orders, as follows from the multi-cover skein relation.
[*Ekholm-Kucharski-Longhi, arXiv: 1910.06193*]

Local equivalence

Verifying systematically conditions from the above theorem, we can identify all equivalent quivers associated to a given knot.



$\mathbf{4_1}$



$$\begin{bmatrix} 0 & -1 & -1 & 0 & 0 \\ -1 & -2 & -2 & -1 & \boxed{0} \\ -1 & -2 & -1 & \boxed{-1} & 0 \\ 0 & -1 & \boxed{-1} & 1 & 1 \\ 0 & \boxed{0} & 0 & 1 & 2 \end{bmatrix} \quad \circ\!\!-\!\!-\!\!-\!\!-\!\!\bullet \quad \begin{bmatrix} 0 & -1 & -1 & 0 & 0 \\ -1 & -2 & -2 & -1 & -1 \\ -1 & -2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix}$$

$$\lambda_2\lambda_5 = \lambda_3\lambda_4$$

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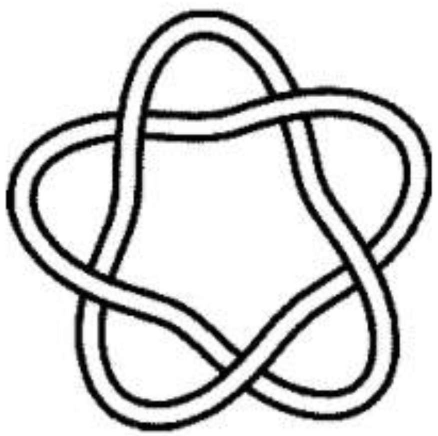
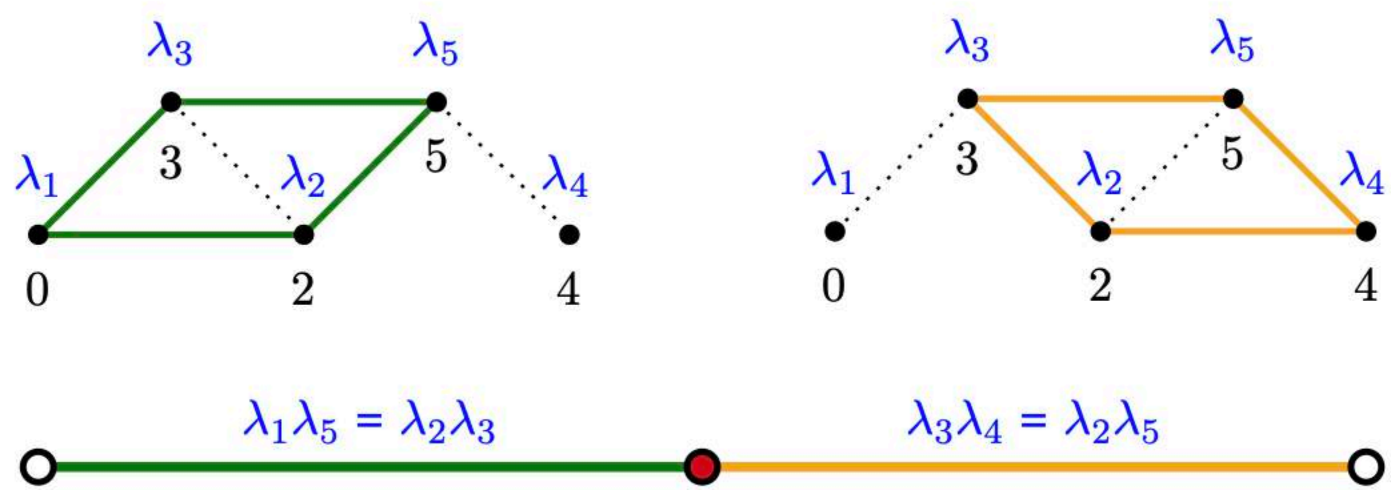


5_1



$$\begin{bmatrix} 0 & 1 & 1 & 3 & \boxed{2} \\ 1 & 2 & \boxed{3} & 3 & 3 \\ 1 & \boxed{3} & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 & 4 \\ \boxed{2} & 3 & 4 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 & 4 \\ 3 & 3 & 4 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 2 & 2 & 3 & \boxed{4} \\ 1 & 2 & 3 & \boxed{3} & 4 \\ 3 & 3 & \boxed{3} & 4 & 4 \\ 3 & \boxed{4} & 4 & 4 & 5 \end{bmatrix}$$

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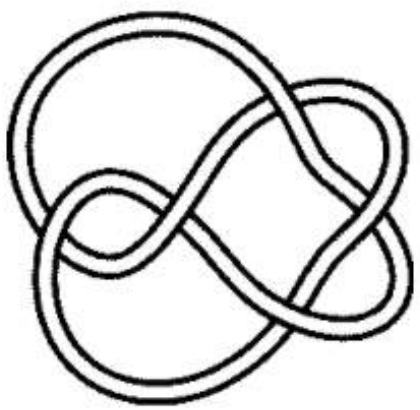
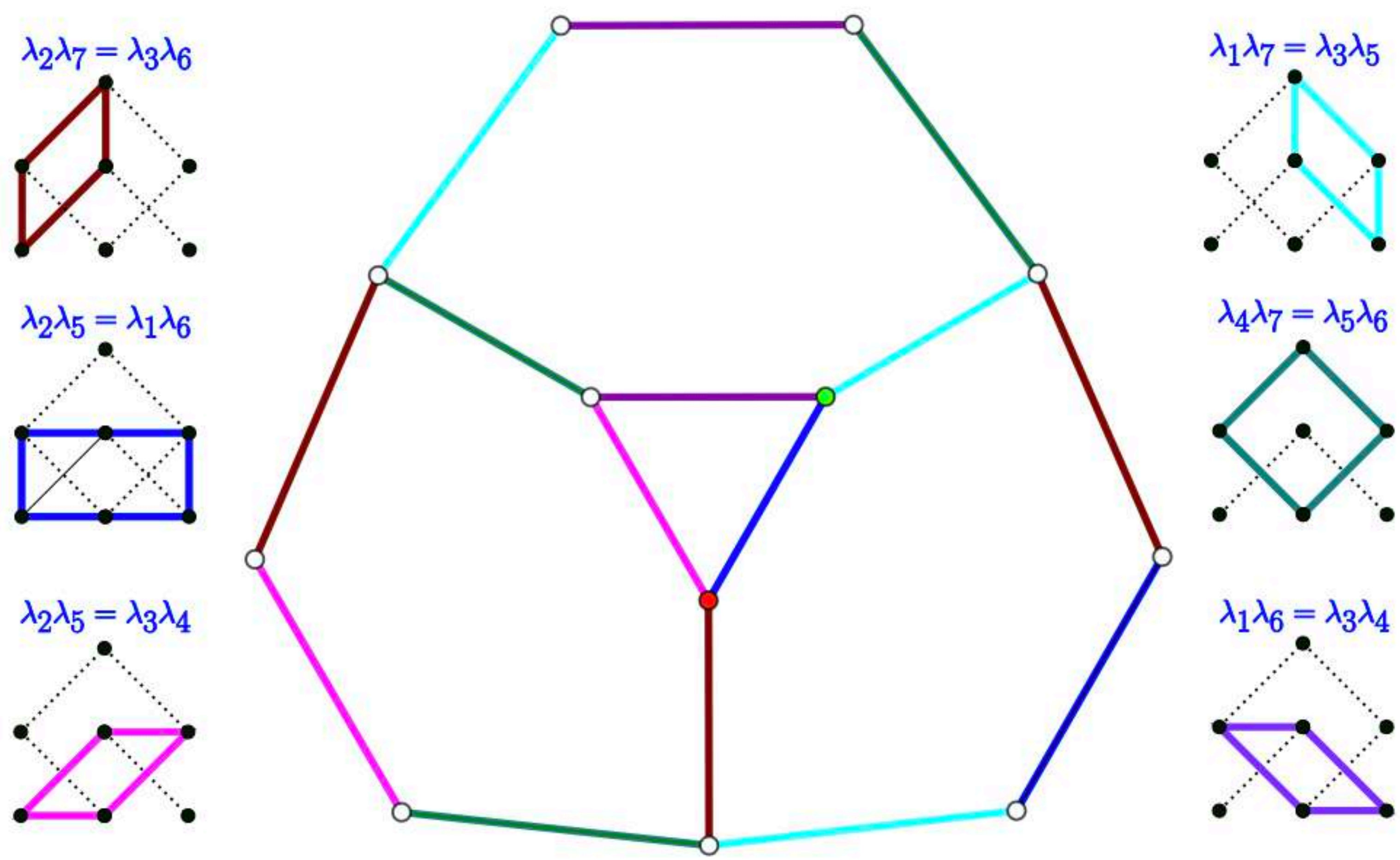


$\mathbf{5_2}$



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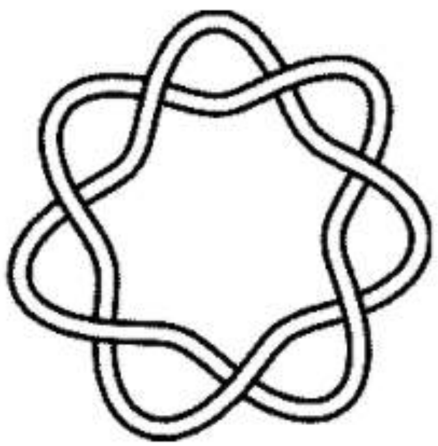
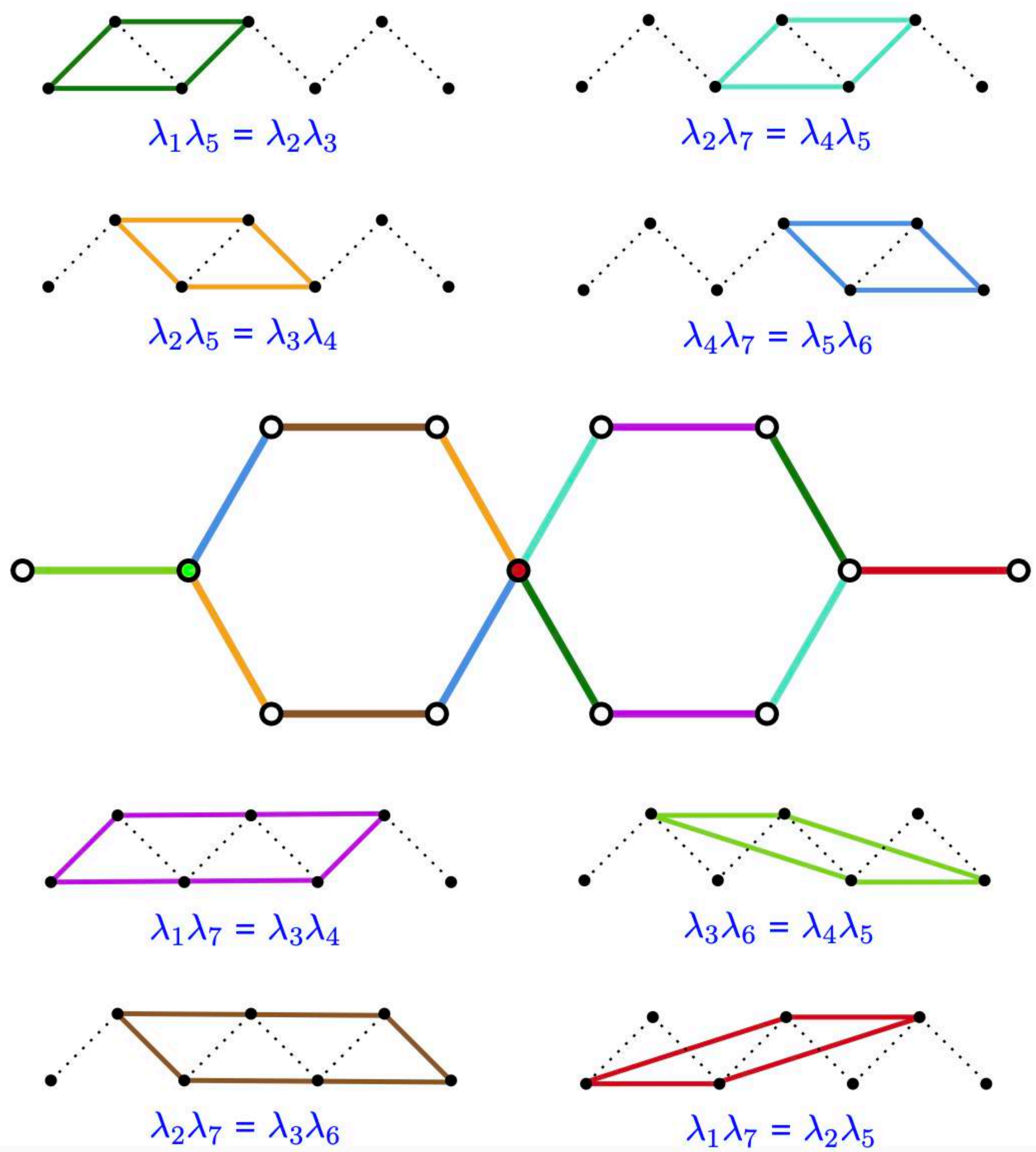


7_1



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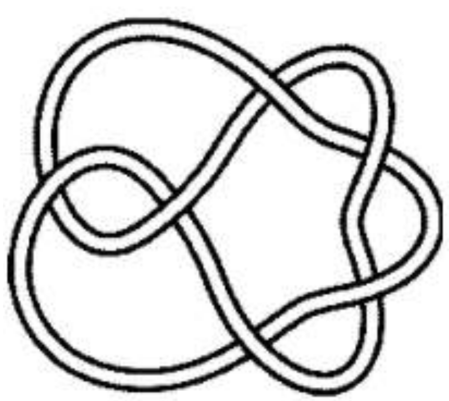
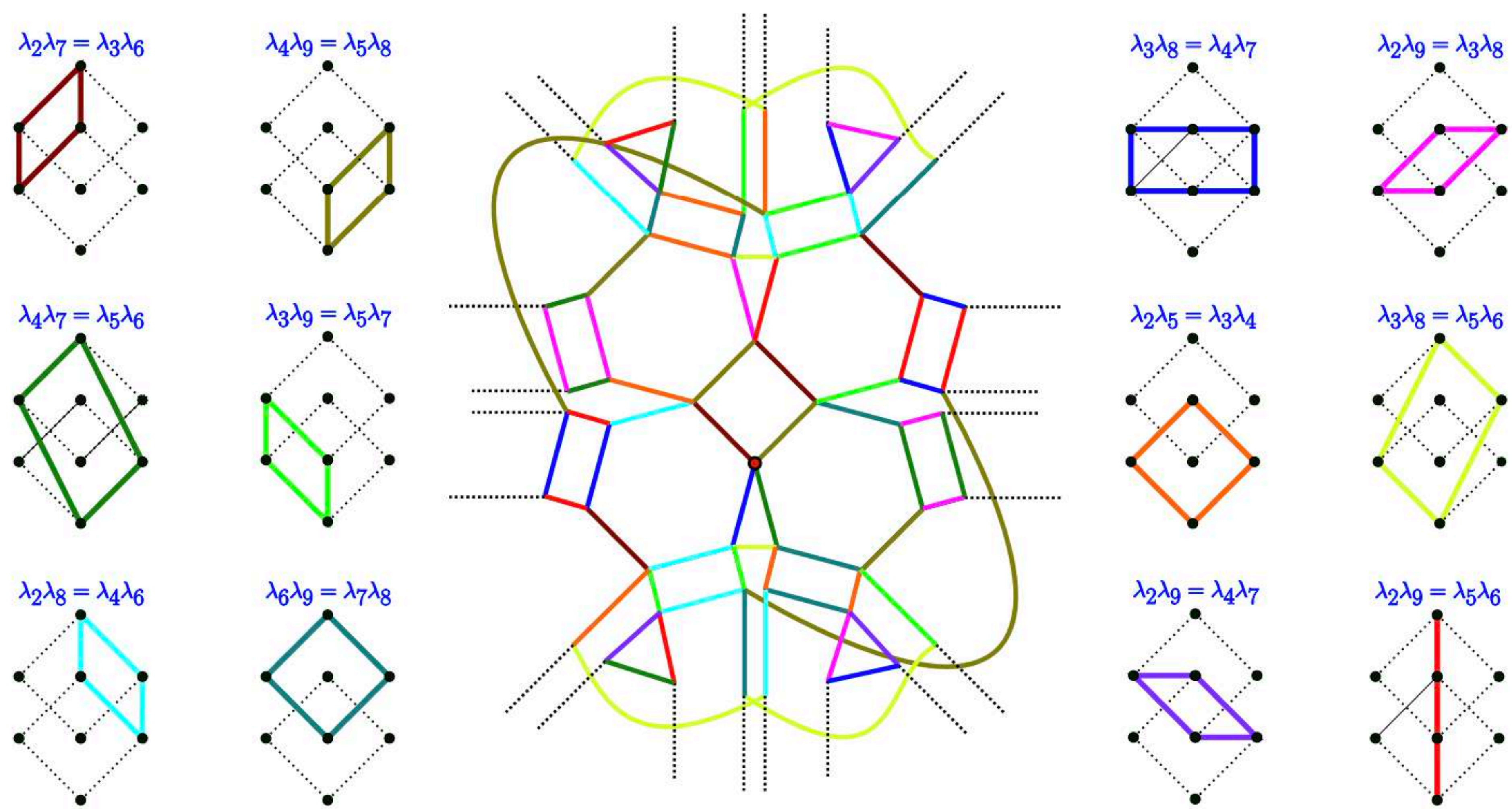


6_1



Global structure

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Instead of analyzing quiver matrices, consider the structure of quiver generating series. We find that in general it takes form

$$P_K(x,a,q,t) = \sum_{\check{d}_1,\ldots,\check{d}_{m-n}\geq 0} (-q)^{\sum_{i,j}\check{C}_{ij}\check{d}_i\check{d}_j} \frac{\check{x}_1^{\check{d}_1}\cdots\check{x}_{m-n}^{\check{d}_{m-n}}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_{m-n}}} \Pi_{\check{d}_1,\ldots,\check{d}_n}\Big|_{\check{x}_i=x\check{\lambda}_i}$$

where $\check{C}$ is a matrix of a subquiver, and the last piece takes form

$$\frac{\Pi_{\check{d}_1,\ldots,\check{d}_n}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_n}} = \sum_{\check{d}_1=\alpha_1+\beta_1}\cdots\sum_{\check{d}_n=\alpha_n+\beta_n} \frac{(-q)^{2\sum_{i<j}\beta_i\alpha_j+\pi_2(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)}\kappa^{\beta_1+\ldots+\beta_n}}{(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\beta_1}\cdots(q^2;q^2)_{\alpha_n}(q^2;q^2)_{\beta_n}}$$

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$$P_K(x,a,q,t) = \sum_{\check{d}_1,\ldots,\check{d}_{m-n}\geq 0} (-q)^{\sum_{i,j}\check{C}_{ij}\check{d}_i\check{d}_j} \frac{\check{x}_1^{\check{d}_1}\cdots\check{x}_{m-n}^{\check{d}_{m-n}}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_{m-n}}} \Pi_{\check{d}_1,\ldots,\check{d}_n}\Big|_{\check{x}_i=x\check{\lambda}_i}$$

where $\check{C}$ is a matrix of a subquiver, and the last piece takes form

$$\frac{\Pi_{\check{d}_1,\ldots,\check{d}_n}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_n}} = \sum_{\check{d}_1=\alpha_1+\beta_1}\cdots\sum_{\check{d}_n=\alpha_n+\beta_n} \frac{(-q)^{2\sum_{i<j}\beta_i\alpha_j+\pi_2(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)}\kappa^{\beta_1+\ldots+\beta_n}}{(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\beta_1}\cdots(q^2;q^2)_{\alpha_n}(q^2;q^2)_{\beta_n}}$$

Recall: a permutation σ is determined by a set of its inversions, i.e. a set of all pairs (σ(*i*),σ(*j*)) such that *i*<*j* and σ(*i*)>σ(*j*).

Such a set of inversions is encoded in the term $\sum_{i<j}\beta_i\alpha_j$

Global structure

Instead of analyzing quiver matrices, consider the structure of quiver generating series. We find that in general it takes form

$$P_K(x,a,q,t) = \sum_{\check{d}_1,\ldots,\check{d}_{m-n}\geq 0} (-q)^{\sum_{i,j}\check{C}_{ij}\check{d}_i\check{d}_j} \frac{\check{x}_1^{\check{d}_1}\cdots\check{x}_{m-n}^{\check{d}_{m-n}}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_{m-n}}} \Pi_{\check{d}_1,\ldots,\check{d}_n}\Big|_{\check{x}_i=x\check{\lambda}_i}$$

where $\check{C}$ is a matrix of a subquiver, and the last piece takes form

$$\frac{\Pi_{\check{d}_1,\ldots,\check{d}_n}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_n}} = \sum_{\check{d}_1=\alpha_1+\beta_1}\cdots\sum_{\check{d}_n=\alpha_n+\beta_n} \frac{(-q)^{2\sum_{i<j}\beta_i\alpha_j+\pi_2(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)}\kappa^{\beta_1+\ldots+\beta_n}}{(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\beta_1}\cdots(q^2;q^2)_{\alpha_n}(q^2;q^2)_{\beta_n}}$$

Recall: a permutation σ is determined by a set of its inversions, i.e. a set of all pairs (σ(*i*),σ(*j*)) such that *i*<*j* and σ(*i*)>σ(*j*). Such a set of inversions is encoded in the term $\sum_{i<j}\beta_i\alpha_j$

It follows that various identifications of indices lead to a permutohedron of equivalent quivers!

Global structure

Such structures arise from the following formula for

$$\Pi_{\check{d}_1,\ldots,\check{d}_n} = (\xi;q^2)_{\check{d}_1+\ldots+\check{d}_n}$$

(or its generalizations):

$$\frac{(\xi;q^2)_{\check{d}_1+\ldots+\check{d}_n}}{(q^2;q^2)_{\check{d}_1}\cdots(q^2;q^2)_{\check{d}_n}} = \sum_{\alpha_1+\beta_1=\check{d}_1}\cdots\sum_{\alpha_n+\beta_n=\check{d}_n}(-q)^{\beta_1^2+\ldots+\beta_n^2+2\sum_{i=1}^{n-1}\beta_{i+1}(\check{d}_1+\ldots+\check{d}_i)}\times$$
$$\times\frac{(\xi q^{-1})^{\beta_1+\cdots+\beta_n}}{(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\beta_1}\cdots(q^2;q^2)_{\alpha_n}(q^2;q^2)_{\beta_n}},$$

Global structure

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(or its generalizations):

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$$\times\frac{(\xi q^{-1})^{\beta_1+\cdots+\beta_n}}{(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\beta_1}\cdots(q^2;q^2)_{\alpha_n}(q^2;q^2)_{\beta_n}},$$

Such form of quiver generating functions follows from constraints in the local equivalence theorem.

Global structure

We refer to the subquiver mentioned above as a ``prequiver''.
The full quiver is determined from a sub quiver by permutation and a pair of integers (*k*,*l*), in the operation called ``splitting''.

$$\begin{pmatrix} \check{C}_{ss} & \cdots & \check{C}_{si} & \cdots & \check{C}_{sj} \\ \vdots & \ddots & \vdots & & \vdots \\ \check{C}_{is} & \cdots & \check{C}_{ii} & \cdots & \check{C}_{ij} \\ \vdots & & \vdots & \ddots & \vdots \\ \check{C}_{js} & \cdots & \check{C}_{ji} & \cdots & \check{C}_{jj} \end{pmatrix}$$

$\sigma(i) < \sigma(j)$

$$\begin{pmatrix} \check{C}_{ss} & \cdots & \check{C}_{si} & \check{C}_{si}+h_s & \cdots & \check{C}_{sj} & \check{C}_{sj}+h_s \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \check{C}_{is} & \cdots & \check{C}_{ii} & \check{C}_{ii}+k & \cdots & \check{C}_{ij} & \check{C}_{ij}+k \\ \check{C}_{is}+h_s & \cdots & \check{C}_{ii}+k & \check{C}_{ii}+l & \cdots & \check{C}_{ij}+k+1 & \check{C}_{ij}+l \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \check{C}_{js} & \cdots & \check{C}_{ji} & \check{C}_{ji}+k+1 & \cdots & \check{C}_{jj} & \check{C}_{jj}+k \\ \check{C}_{js}+h_s & \cdots & \check{C}_{ji}+k & \check{C}_{ji}+l & \cdots & \check{C}_{jj}+k & \check{C}_{jj}+l \end{pmatrix}$$

$\sigma(i) > \sigma(j)$

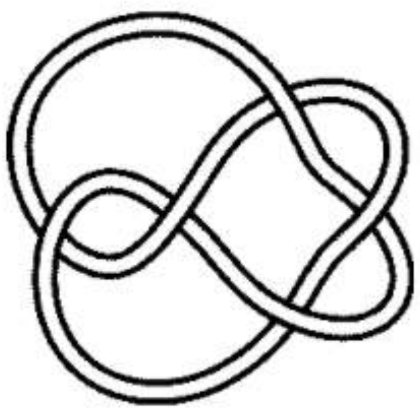
$$\begin{pmatrix} \check{C}_{ss} & \cdots & \check{C}_{si} & \check{C}_{si}+h_s & \cdots & \check{C}_{sj} & \check{C}_{sj}+h_s \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \check{C}_{is} & \cdots & \check{C}_{ii} & \check{C}_{ii}+k & \cdots & \check{C}_{ij} & \check{C}_{ij}+k+1 \\ \check{C}_{is}+h_s & \cdots & \check{C}_{ii}+k & \check{C}_{ii}+l & \cdots & \check{C}_{ij}+k & \check{C}_{ij}+l \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \check{C}_{js} & \cdots & \check{C}_{ji} & \check{C}_{ji}+k & \cdots & \check{C}_{jj} & \check{C}_{jj}+k \\ \check{C}_{js}+h_s & \cdots & \check{C}_{ji}+k+1 & \check{C}_{ji}+l & \cdots & \check{C}_{jj}+k & \check{C}_{jj}+l \end{pmatrix}$$

Global structure

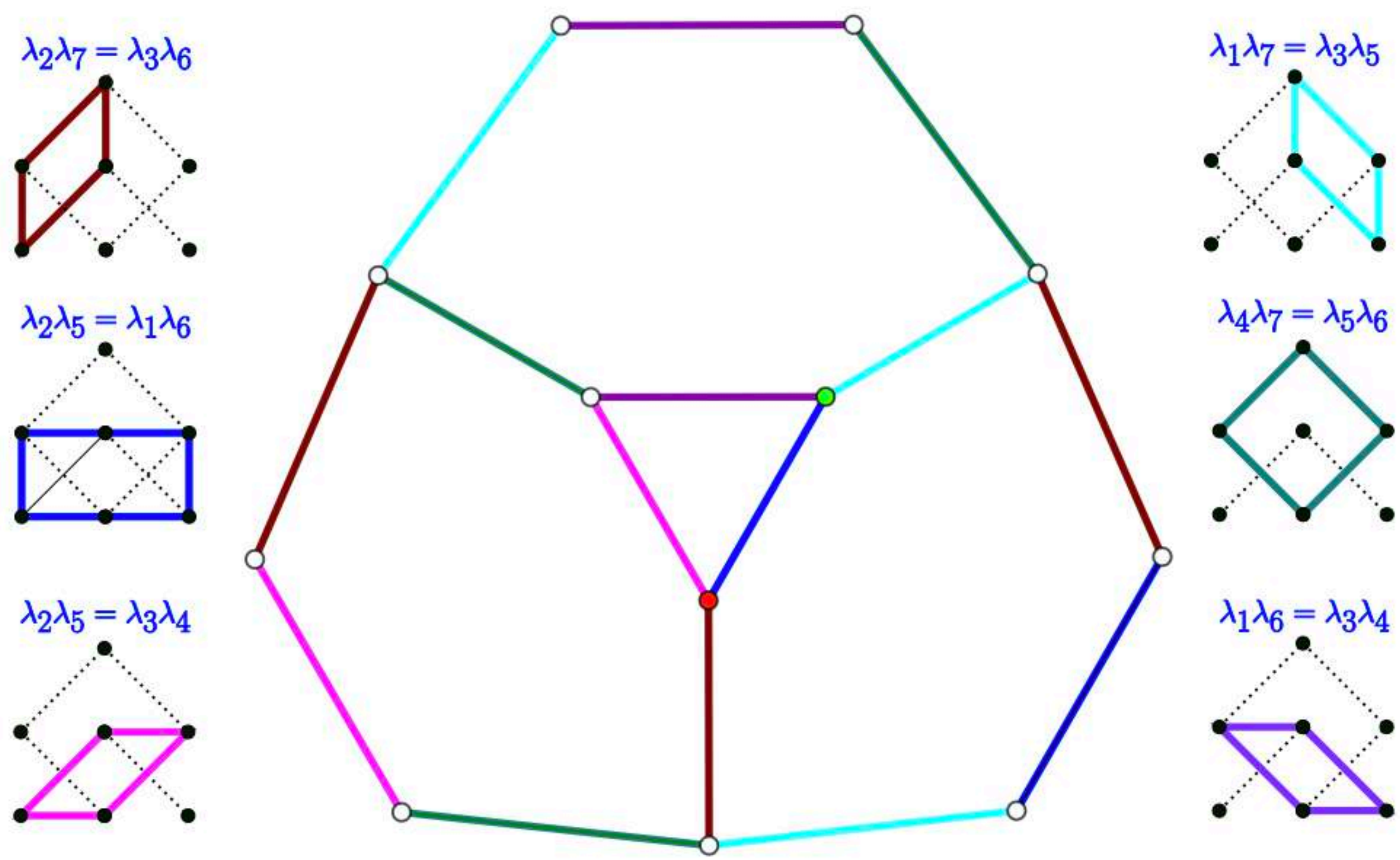
In general, there are several equivalent formulas for a given HOMFLY-PT generating function, and each of them gives rise to one permutohedron. Altogether we obtain a large permutohedron graph.

Global structure – examples

In this case permutohedron graph is made of 3 permutohedra Π_3.

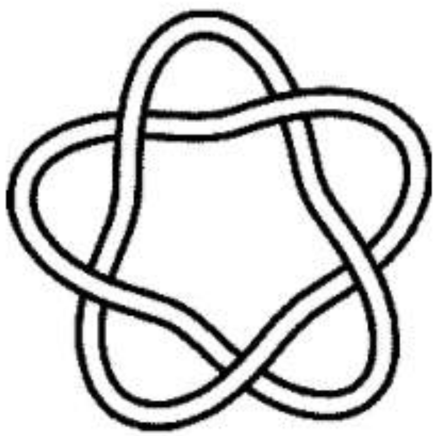


5_2

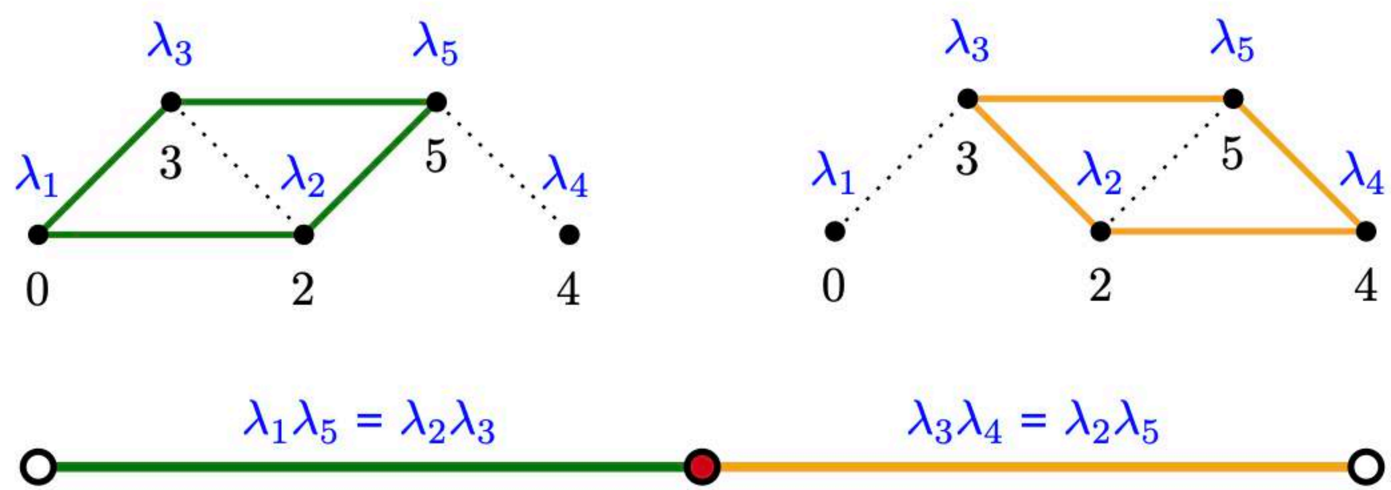


Global structure – examples

For (2,2p+1) torus knots, permutohedron graph is made of two chains of larger and larger permutohedra.



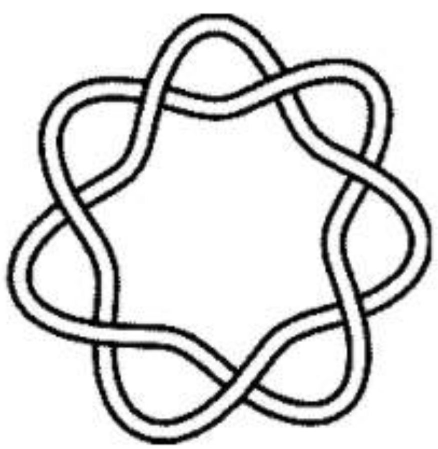
5_1



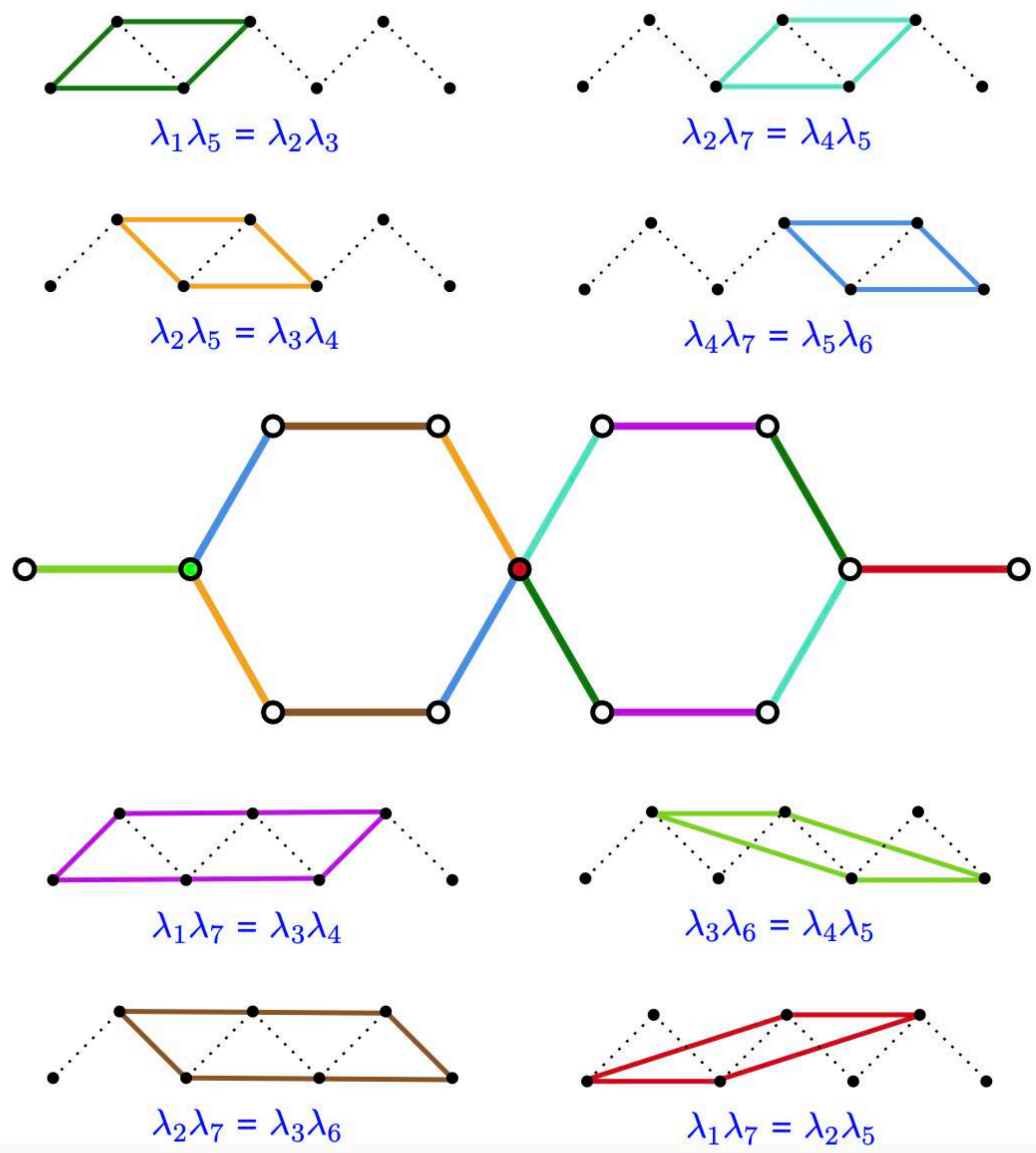
$$\begin{bmatrix} 0 & 1 & 1 & 3 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 3 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 & 4 \\ 2 & 3 & 4 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 & 4 \\ 3 & 3 & 4 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 & 4 \\ 3 & 3 & 3 & 4 & 4 \\ 3 & 4 & 4 & 4 & 5 \end{bmatrix}$$

Global structure – examples

For (2,2p+1) torus knots, permutohedron graph is made of two chains of larger and larger permutohedra.



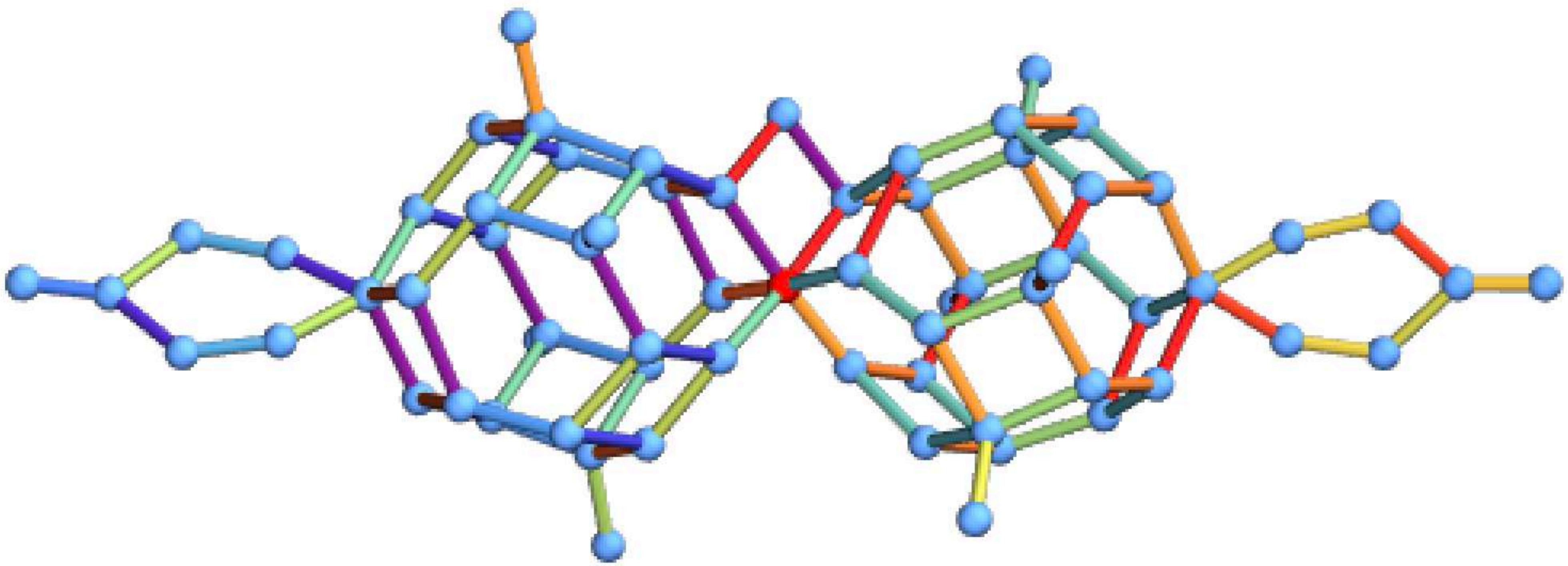
7_1



Global structure – examples

For (2,2p+1) torus knots, permutohedron graph is made of two chains of larger and larger permutohedra.

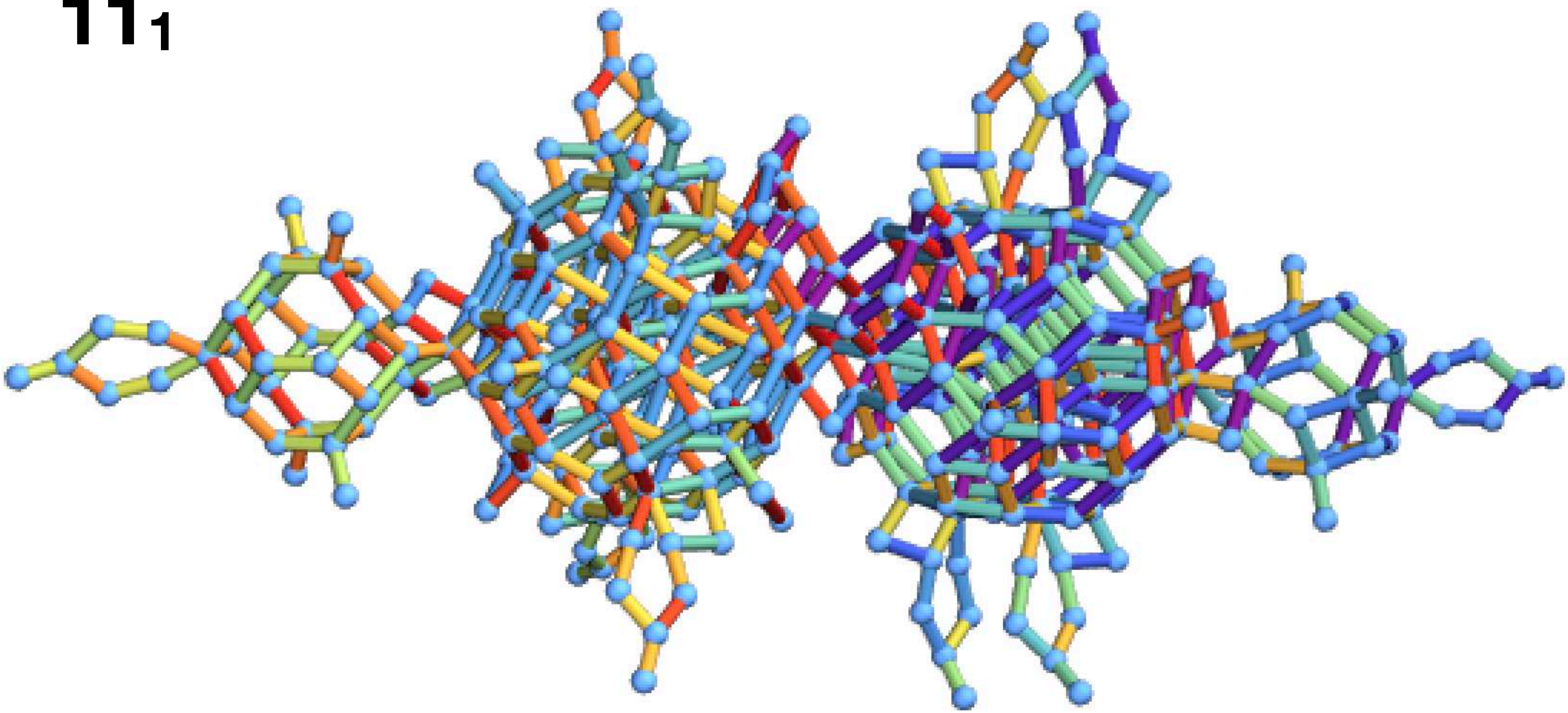
9_1



Global structure – examples

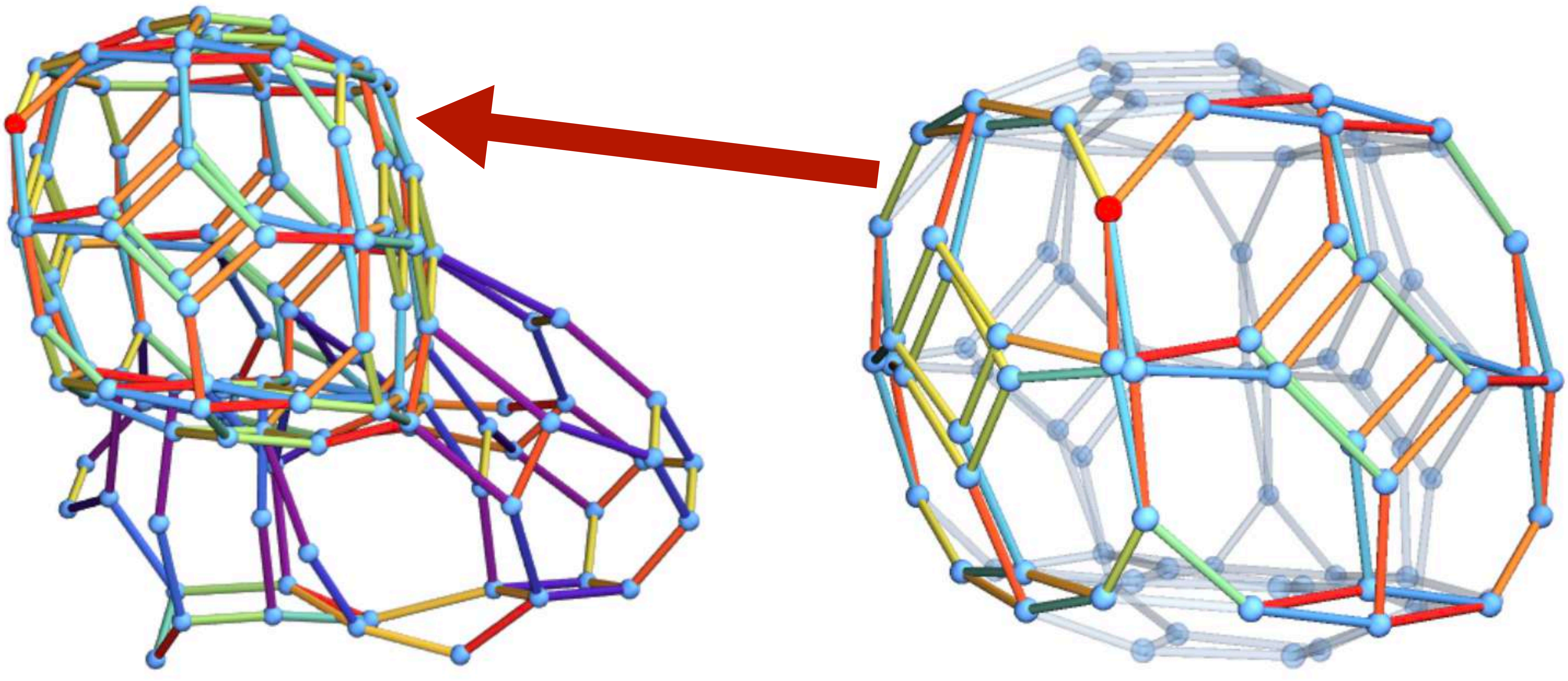
For (2,2p+1) torus knots, permutohedron graph is made of two chains of larger and larger permutohedra.

$\mathbf{11_1}$



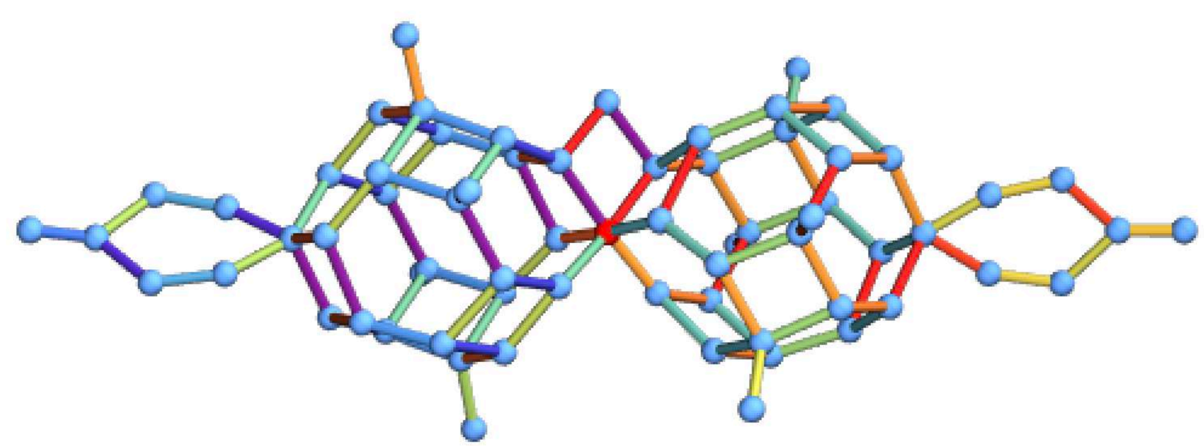
Global structure – examples

6_1



Summary

- Knots-quivers correspondence, motivated by string theory, relates knot theory and quiver representation theory
- It turns out that many quivers may be associated to a given knot
- They are parametrized by vertices of a permutohedron graph
- This indicates some interesting structure of the underlying HOMFLY-PT homology, and of the corresponding LMOV (motivic DT) invariants



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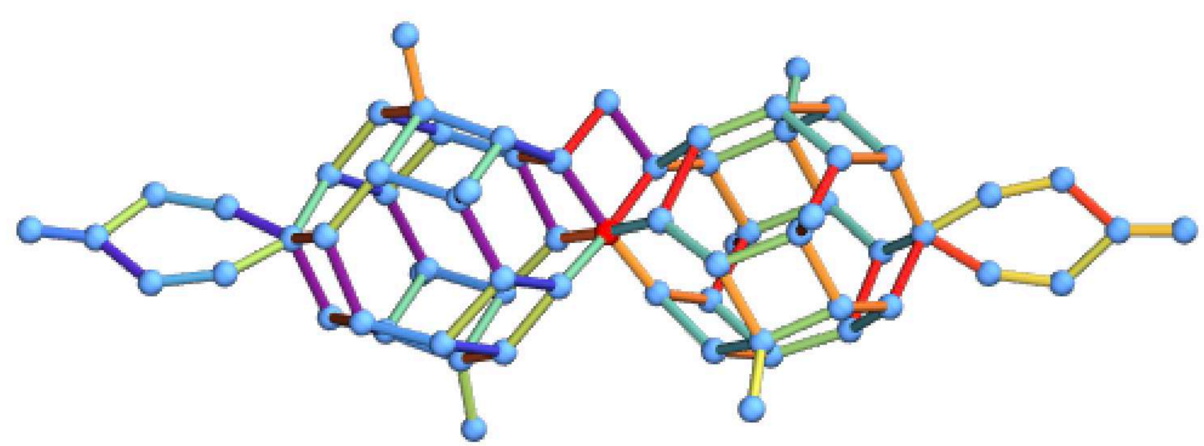
Summary

Future directions and related developments:

- identify permutohedra for rational and arborescent knots (following Stosic-Wedrich, arXiv: 1711.03333, 2004.10837)
- develop open topological string interpretation (following Ekholm-Kucharski-Longhi, arXiv: 1811.03110, 1910.06193)
- conduct analogous analysis for other underlying toric Calabi-Yau manifolds (following Kimura-Panfil-Sugimoto-PS , arXiv: 1811.03556, 2011.06783)



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