Super J-holomorphic curves

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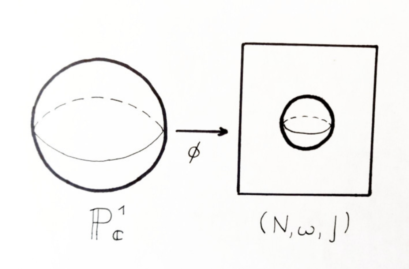
J-holomorphic curves[1]

- A J-holomorphic curve $\phi\colon \Sigma \to N$ is a map from a Riemann surface Σ to an almost Kähler manifold (N, ω, J) such that

$$\overline{\partial}_J \phi = \frac{1}{2}\left(d\phi + J\, d\phi\, \mathrm{I}\right) = 0 \in \Gamma\left(T^\vee \Sigma \otimes \phi^* TN\right)^{0,1}$$



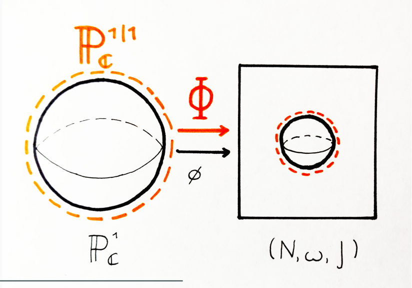
[1] McDuff and Salamon (2012). *J-holomorphic curves and symplectic topology*.

J-holomorphic curves

- J-holomorphic curves are absolute minimizers of the Dirichlet action.
- Under certain conditions the moduli space $M^*_p(A)$ of J-holomorphic curves and $[\mathrm{im}\,\phi] = A \in H_2(N, \mathbb{Z})$ is a manifold.
- There is a compactification $\overline{M}_{p,k}(A)$ via stable maps.
- Gromov–Witten invariants of N can be constructed as certain integrals over $\overline{M}_{p,k}(A)$.

- A *super* J-holomorphic curve $\Phi\colon M \to N$ is a map from a *super* Riemann surface M such that

$$\overline{D}_J\Phi = \frac{1}{2}\left(d\Phi + J\, d\Phi\, \mathrm{I}\right)|_{\mathcal{D}} = 0 \in \Gamma\left(\mathcal{D}^\vee \otimes \Phi^* TN\right)^{0,1}$$



[2] Keßler, Sheshmani, and Yau (2021). "Super J-holomorphic Curves: Construction of the Moduli Space."

Super *J*-holomorphic curves of genus zero

- The differential equations of super *J*-holomorphic curves couple the Cauchy–Riemann equations of *J*-holomorphic curves with a Dirac equation for spinors.
- Super *J*-holomorphic curves are critical points of the superconformal action or spinning string action.
- Under certain conditions the moduli space $\mathcal{M}(A)$ of *J*-holomorphic curves of genus 0 and $[\mathrm{im}\,\phi] = A \in H_2(N, \mathbb{Z})$ is a *supermanifold*.
- There is a compactification $\overline{\mathcal{M}}_{0,k}(A)$ via *super* stable maps.
- *Super* Gromov–Witten invariants?

Outline

Super Riemann Surfaces

Super J-holomorphic curves

Moduli Space of super J-holomorphic curves

Super Stable Maps

Super Riemann Surfaces

Graßmann algebras

- Think of differential forms with exterior product $\wedge$.
- The exterior algebra of a vector space V is defined as $\bigwedge(V) = {\mathcal{T}(V)}\big/_{\langle v \otimes v \rangle}$.
- $\mathbb{Z}_2$-grading: $\bigwedge(V) = \bigwedge_0(V) \oplus \bigwedge_1(V)$.
- *supercommutative* product: $a \cdot b = (-1)^{p(a)p(b)} b \cdot a$.
- For $V = \mathbb{R}^n$ we denote a basis by η^α and then any element $a \in \bigwedge(\mathbb{R}^n)$ can be written

$$a = a_0 + \eta^\alpha{}_\alpha a + \eta^\alpha \eta^\beta{}_{\alpha\beta} a + \ldots + \eta^1 \cdot \cdots \cdot \eta^n{}_{1 \ldots n} a.$$

- Homomorphisms of Graßmann algebras preserve the $\mathbb{Z}_2$-grading.

Super geometry was developed in the 1980s to provide mathematical tools for supersymmetric field theories.[3]

The building block for supergeometry is the ringed space $\mathbb{R}^{m|n} = (\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^{m|n}})$, where

$$\mathcal{O}_{\mathbb{R}^{m|n}} = C^\infty(\mathbb{R}^m, \mathbb{R}) \otimes \bigwedge(\mathbb{R}^n).$$

- even coordinates $x^1, \ldots, x^m$, odd coordinates $\eta^1, \ldots, \eta^n$
- general function on $\mathbb{R}^{2|2}$:
  $f(x, \eta) = {}_0f(x) + \eta^\mu \, {}_\mu f(x) + \eta^1\eta^2 \, {}_{12}f(x)$
- Supermanifolds are locally isomorphic to $\mathbb{R}^{m|n}$.
- Maps of supermanifolds are maps of ringed spaces.
- Any manifold is a supermanifold of odd dimension zero.

[3] Leites (1980). "Introduction to the theory of supermanifolds."

Let (x^a, η^α) be coordinates on $\mathbb{R}^{m|n}$. Tangent vector fields on $\mathbb{R}^{m|n}$ are derivations on the functions on $\mathbb{R}^{m|n}$. They can be written as a linear combination of the partial derivatives $\partial_{x^a}, \partial_{\eta^\alpha}$.

$$X = X^a \partial_{x^a} + X^\alpha \partial_{\eta^\alpha}$$

Similarly: vector bundles, Lie groups, principal bundles, connections and (almost) complex structures...

Let (y^a, θ^α) be coordinates on $\mathbb{R}^{p|q}$ and (x^b, η^β) be coordinates on $\mathbb{R}^{m|n}$. A map $\Phi\colon \mathbb{R}^{p|q} \to \mathbb{R}^{m|n}$ is completely determined by the image of the coordinate functions:

$$\begin{aligned} \Phi^\# x^b &= {}_0f^b(y) && + \theta^\mu\theta^\nu\, {}_{\nu\mu}f^b(y) + \cdots \\ \Phi^\# \eta^\beta &= \qquad \theta^\mu\, {}_\mu f^\beta(y) && + \cdots \end{aligned}$$

Families of supermanifolds

Let (y^a, θ^α) be coordinates on $\mathbb{R}^{p|q}$ and (x^b, η^β) be coordinates on $\mathbb{R}^{m|n}$. A map $\Phi\colon \mathbb{R}^{p|q} \to \mathbb{R}^{m|n}$ is completely determined by the image of the coordinate functions:

$$\Phi^\# x^b = {}_0f^b(y) + \theta^\mu\, {}_\mu f^b(y) + \theta^\mu\theta^\nu\, {}_{\nu\mu}f^b(y) + \cdots$$
$$\Phi^\# \eta^\beta = {}_0f^\beta(y) + \theta^\mu\, {}_\mu f^\beta(y) + \theta^\mu\theta^\nu\, {}_{\nu\mu}f^\beta(y) + \cdots$$

For full θ-expansion we need families of supermanifolds + base change. Here: Submersions. That is, we actually consider maps $\Phi\colon \mathbb{R}^{p|q} \times B \to \mathbb{R}^{m|n}$.

Projective Superspace $\mathbb{P}_{\mathbb{C}}^{1|1}$

- The complex projective superspace of dimension $1|1$ is a complex supermanifold given by two charts isomorphic to $\mathbb{C}^{1|1}$ with coordinates (z_1, θ_1) and (z_2, θ_2) such that

$$z_2 = \frac{1}{z_1}, \qquad \theta_2 = \frac{\theta_1}{z_1}.$$

- Alternatively $\mathbb{P}_{\mathbb{C}}^{1|1} = \mathrm{Split}_{\mathbb{C}} S = \left(\mathbb{P}_{\mathbb{C}}^1, \bigwedge_{\mathbb{C}}(\mathcal{H}(S))\right)$, where $S \to \mathbb{P}_{\mathbb{C}}^1$ is the spinor line bundle, that is $S \otimes S = T\mathbb{P}_{\mathbb{C}}^1$.

Definition
A super Riemann surface is a complex $1|1$-dimensional supermanifold M with an odd holomorphic distribution $\mathcal{D} \subset TM$, such that $\frac{1}{2}[\cdot,\cdot] \colon \mathcal{D} \otimes_{\mathbb{C}} \mathcal{D} \simeq {}^{TM}\!/\!_{\mathcal{D}}$.

$$0 \longrightarrow \mathcal{D} \longrightarrow TM \longrightarrow {}^{TM}\!/\!_{\mathcal{D}} = \mathcal{D} \otimes \mathcal{D} \longrightarrow 0$$

[4] LeBrun and Rothstein (1988). "Moduli of super Riemann surfaces."

Local structure of SRS

- Let (z, θ) be the standard coordinates on $\mathbb{C}^{1|1}$ and define $\mathcal{D} \subset T\mathbb{C}^{1|1}$ by $\mathcal{D} = \langle \partial_\theta + \theta \partial_z \rangle$. Then $\mathcal{D} \otimes_{\mathbb{C}} \mathcal{D} \simeq {}^{TM}\!/_{\mathcal{D}}$ by

$$[\partial_\theta + \theta\partial_z, \partial_\theta + \theta\partial_z] = 2\partial_z.$$

- Local uniformization: Every super Riemann surfaces is locally isomorphic to $\mathbb{C}^{1|1}$ with its standard super Riemann surface structure.
- A holomorphic map $\Phi \colon \mathbb{C}^{1|1} \to \mathbb{C}$ given by $\Phi(z, \theta) = \varphi(z) + \theta\psi(z)$ satisfies $D\Phi = \psi(z) + \theta\partial_z\varphi$.

Split Super Riemann Surfaces

- $\mathbb{P}^{1|1}_{\mathbb{C}}$ is a super Riemann surface with $\mathcal{D}$ generated by $\partial_{\theta_1} + \theta_1 \partial_{z_1}$ and $\partial_{\theta_2} - \theta_2 \partial_{z_2}$.
- By uniformization of super Riemann surfaces, $\mathbb{P}^{1|1}_{\mathbb{C}}$ is the only super Riemann surface of genus zero.[5]
- More generally, for any Riemann surface Σ and spinor bundle $S \to \Sigma$ the supermanifold $\mathrm{Split}\, S$ carries a canonical super Riemann surface structure.

[5] Crane and Rabin (1988). "Super Riemann surfaces: Uniformization and Teichmüller Theory."

Odd deformations

Let M_{red} be the reduced manifold of a super Riemann surface M (over B) and set $|M| = M_{red} \times B$. Pick a map $i\colon |M| \to M$ which is the identity on the topological spaces.

- The super Riemannn surface M is completely determined by a Riemannian metric g, a spinor bundle S and a gravitino $\chi \in \Gamma\left(T^\vee|M| \otimes S\right)$ on $|M|$.[6]

$$\begin{tikzcd} 0 \arrow[r] & S = i^*\mathcal{D} \arrow[r] & i^*TM \arrow[r] & i^*TM/\mathcal{D} = T|M| \arrow[r] \arrow[l, bend right, "di"'] \arrow[ll, bend left, "\chi"] & 0 \end{tikzcd}$$

[6] Keßler (2019). *Supergeometry, Super Riemann Surfaces and the Superconformal Action Functional.*

Approaches to moduli spaces of SRS

- *Deligne, 1987* : Deformation Theory
- *LeBrun–Rothstein, 1988:* Moduli of marked SRS as "canonical super orbifolds"
- *Crane–Rabin, 1988:* Uniformization of SRS
- *Sachse, 2009:* $\{M \text{ SRS}\}/\mathrm{Diff}_0\, M$
- *Donagi–Witten 2012:* Super moduli space is not projected
- *D'Hoker–Phong*, 1988 / *Keßler* 2019: Metrics and Gravitinos

Super J-holomorphic curves

Definition
Let M be a super Riemann surface and (N, ω, J) an almost Kähler manifold. A map $\Phi\colon M \to N$ is called a super J-holomorphic curve[7] if

$$\overline{D}_J\Phi = \frac{1}{2}\left(d\Phi + J\, d\Phi\, \mathrm{I}\right)\big|_{\mathcal{D}} = 0.$$

[7] Keßler, Sheshmani, and Yau (2021). "Super J-holomorphic Curves: Construction of the Moduli Space."

Let (z, θ) be the superconformal coordinates on $\mathbb{C}^{1|1}$, λ^σ coordinates of B and Z^b complex coordinates on $\mathbb{C}^n$. Any smooth map $\Phi\colon \mathbb{C}^{1|1} \times B \to \mathbb{C}^n$ can be written in coordinates as

$$\Phi^b = \varphi^b(z, \bar{z}, \lambda) + \theta\psi^b(z, \bar{z}, \lambda) + \overline{\theta\psi}^b(z, \bar{z}, \lambda) + \theta\overline{\theta}F^b(z, \bar{z}, \lambda),$$

The map Φ is super J-holomorphic if

$$\begin{aligned}\left(\partial_{\overline{\theta}} + \overline{\theta}\partial_{\overline{z}}\right)\Phi^b &= \overline{\psi}^b(z, \bar{z}, \lambda) - \theta F^b(z, \bar{z}, \lambda) \\ &\quad + \overline{\theta}\partial_{\overline{z}}\varphi^b(z, \bar{z}, \lambda) - \theta\overline{\theta}\partial_{\overline{z}}\psi^b(z, \bar{z}, \lambda) \\ &= 0\end{aligned}$$

- If N is Kähler the map Φ is holomorphic.
- $\Phi_{red}\colon M_{red} \to N$ is a J-holomorphic curve.

Super J-holomorphic curves in component fields

For a super Riemann surface M, $i\colon |M| \to M$ and a map $\Phi\colon M \to N$ define

$$\varphi = \Phi \circ i\colon |M| \to N$$
$$\psi = i^* \, d\Phi|_{\mathcal{D}} \in \Gamma\left(S^\vee \otimes \varphi^* TN\right)$$
$$F = i^* \Delta^{\mathcal{D}} \Phi \in \Gamma\left(\varphi^* TN\right)$$

Theorem
The map Φ is a super J-holomorphic curve if and only if

$$\overline{\partial}_J \varphi + \langle Q\chi, \psi \rangle = 0, \qquad (1 + \mathrm{I} \otimes J)\,\psi = 0$$
$$F = 0, \qquad \not{D}\psi - 2\,\langle \vee Q\chi, d\varphi \rangle + \|Q\chi\|^2 \psi = 0$$

Here we have assumed for simplicity that N is Kähler.

Super J-holomorphic curves are super harmonic

Proposition

Any super J-holomorphic curve $\Phi\colon M \to N$ is a critical point of the superconformal action

$$
\begin{aligned}
A(M,\Phi) = & \int_M \| d\Phi|_{\mathcal{D}}\|^2 [dvol] \\
= & \int_{|M|} \Big(\|d\varphi\|^2_{g^\vee \otimes \varphi^* n} + g_S^\vee \otimes \varphi^* n \left(\not{D}\psi, \psi\right) - \|F\|^2_{\varphi^* n} \\
& + 4 g^\vee \otimes \varphi^* n \left(d\varphi, \langle Q\chi, \psi\rangle\right) + \|Q\chi\|^2_{g^\vee \otimes g_S} \|\psi\|^2_{g_S^\vee \otimes \varphi^* n} \\
& - \frac{1}{6} g_S^\vee \otimes \varphi^* n \left(SR^N(\psi), \psi\right) \Big) \, dvol_g
\end{aligned}
$$

Moduli Space of super J-holomorphic curves

B-points of $\mathbb{R}^{m|n}$

Let $B = \mathbb{R}^{0|s}$. A B-point of $\mathbb{R}^{m|n}$ is a map $p\colon B \to \mathbb{R}^{m|n}$.

$$p^{\#}x^{a} = p^{a} \qquad\qquad p^{\#}\eta^{\alpha} = p^{\alpha}$$

The set of B-points of $\mathbb{R}^{m|n}$ is given by

$$\underline{\mathbb{R}^{m|n}}(B) = (\mathcal{O}_B)_0^m \oplus (\mathcal{O}_B)_1^n = \left(\bigwedge\nolimits_s\right)_0^m \oplus \left(\bigwedge\nolimits_s\right)_1^n$$

More generally for a supermanifold M we obtain a functor

$$\begin{aligned}\underline{M}\colon \mathrm{SPoint}^{op} &\to \mathrm{Man}\\ B &\mapsto \underline{M}(B)\end{aligned}$$

that contains the full information on M[8].

[8] Molotkov (2010). "Infinite Dimensional and Colored Supermanifolds."

Space of all maps

Let $\mathcal{H}$ be the infinite-dimensional supermanifold such that

$$\underline{\mathcal{H}}(B) = \{\Phi\colon M \times B \to N\}.$$

Charts can be constructed using the exponential map on the target N.

Let furthermore $\mathcal{E} \to \mathcal{H}$ be the vector bundle such that above Φ we have

$$\underline{\mathcal{E}_\Phi}(B) = \Gamma\left(\mathcal{D}^\vee \otimes \Phi^* TN\right)^{0,1} = \mathrm{codomain}\, \overline{D}_J \Phi$$

Then $\mathcal{S} = \overline{D}_J\colon \mathcal{H} \to \mathcal{E}$ is a section of $\mathcal{E}$ and $\mathcal{S}^{-1}(0)$ is the space of super J-holomorphic curves.

Fix $A \in H_2(N,\mathbb{Z})$ and let $j\colon \mathcal{M}(A) \to \mathcal{H}$ be the embedding of

$$\underline{\mathcal{M}(A)}(B) = \{\Phi \in \underline{\mathcal{H}}(B) \mid \mathcal{S}(\Phi) = 0 \text{ and } [\operatorname{im}\Phi] = A\}.$$

$$j^* \ker d\mathcal{S} \hookrightarrow j^* T\mathcal{H} \xrightarrow{j^*\, d\mathcal{S}} j^* \mathcal{S}^* \mathcal{E} \twoheadrightarrow j^* \operatorname{Coker} d\mathcal{S}$$

$$\downarrow$$

$$\mathcal{M}(A)$$

By the Theorem of Atiyah–Singer applied to the linearizations of $\overline{\partial}_J$ and $\not{D}$:

$$\operatorname{rk}\ker d\mathcal{S} - \operatorname{rk}\operatorname{Coker} d\mathcal{S} = 2n(1-p) + 2\langle c_1(TN), A\rangle \, |2 \langle c_1(TN), A\rangle \, .$$

Moduli space of super J-holomorphic curves

Theorem

Fix a closed compact super Riemann surface M over $\mathbb{R}^{0|0}$ of genus p, an almost Kähler manifold N and $A \in H_2(N)$. If $\mathcal{S}$ is transversal to the zero section, i.e. $d\mathcal{S}$ is surjective, $\mathcal{M}(A)$ is a supermanifold of dimension

$$2n(1-p) + 2\langle c_1(TN), A\rangle \,|2\langle c_1(TN), A\rangle\,.$$

Sketch of Proof

Idea: Use implicit function theorem around a given J-holomorphic curve to obtain local charts for $\mathcal{M}(A)$.

If $\mathcal{S}$ is transversal to the zero section at Φ:

- Complete locally around Φ to Sobolev spaces $\mathcal{E}^{k,p} \to \mathcal{H}^{k,p}$
- Apply Banach space implicit function theorem to obtain a local chart for $\left(\mathcal{S}^{k,p}\right)^{-1}(0)$.
- Show by elliptic regularity that preimages of zero are smooth, that is, in $\mathcal{S}^{-1}(0)$.

Sketch of Proof: Manifold structures

Assume that $\Phi\colon M \to N$ has component fields $\varphi\colon M_{red} \to N$, $\psi = 0$ and $F = 0$. Let $\Phi_B = \Phi \times \mathrm{id}_B\colon M \times B \to N$.

$$\begin{array}{ccc}
\underline{\mathcal{E}}(B) & \xleftarrow{\ \mathrm{id}_{\mathcal{D}} \otimes P^{\overline{\nabla}}_{\exp_{\Phi_B}}\ } & \underline{U}_\Phi(B) \times \Gamma\left(\mathcal{D}^\vee \otimes \Phi_B^* TN\right)_0^{0,1} \\
{\scriptstyle \underline{\mathcal{S}}(B)} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle (\mathrm{id}_{\underline{U}_\Phi(B)}, \underline{\mathcal{F}_\Phi(B)})} \\
\underline{\mathcal{H}}(B) & \xleftarrow{\ \exp_{\Phi_B}\ } & \underline{U}_\Phi(B) \subset \Gamma\left(\Phi_B^* TN\right)_0
\end{array}$$

$$\begin{aligned}
\Gamma\left(\Phi_B^* TN\right)_0 \cong{} & \Gamma\left(\varphi^* TN\right) \otimes \left(\mathcal{O}_B\right)_0 \oplus \Gamma\left(S^\vee \otimes \varphi^* TN\right) \otimes \left(\mathcal{O}_B\right)_1 \\
& \oplus \Gamma\left(\varphi^* TN\right) \otimes \left(\mathcal{O}_B\right)_0 \\
\Gamma\left(\mathcal{D}^\vee \otimes \Phi_B^* TN\right)_0^{0,1} \cong{} & \Gamma\left(S^\vee \otimes \varphi^* TN\right)^{0,1} \otimes \left(\mathcal{O}_B\right)_1 \\
& \oplus \left(\Gamma\left(\varphi^* TN\right) \oplus \Gamma\left(T^\vee M_{red} \otimes \varphi^* TN\right)^{0,1}\right) \otimes \left(\mathcal{O}_B\right)_0 \\
& \oplus \Gamma\left(S^\vee \otimes \varphi^* TN\right)^{0,1} \otimes \left(\mathcal{O}_B\right)_1
\end{aligned}$$

$\underline{\mathcal{S}}$ is transversal to the zero section if the differential of the map

$$\begin{aligned}\underline{\mathcal{F}_\Phi}(C)\colon \Gamma\left(\Phi_C^* TN\right)_0 &\to \Gamma\left(\mathcal{D}^\vee \otimes \Phi_C^* TN\right)_0^{0,1}\\ X &\mapsto \left(\mathrm{id}_{\mathcal{D}^\vee} \otimes P^{\overline{\nabla}}_{\exp_{\Phi_C} tX}\right)^{-1} \overline{D}_J \exp_{\Phi_C} X\end{aligned}$$

is surjective. In component fields the differential is given by

$$\begin{aligned}d\underline{\mathcal{F}_\Phi}(C)\colon \Gamma\left(\Phi_C^* TN\right)_0 &\to \Gamma\left(\mathcal{D}^\vee \otimes \Phi_C^* TN\right)_0^{0,1}\\ (\xi, \zeta, \sigma) &\mapsto (\zeta^{0,1}, \sigma, (1 + \mathrm{I} \otimes J)\nabla\xi, (1 + \mathrm{I} \otimes J)\not{D}\zeta^{1,0})\end{aligned}$$

Note that $(1 + \mathrm{I} \otimes J)\nabla\xi$ and $(1 + \mathrm{I} \otimes J)\not{D}\zeta^{1,0}$ are $(\mathcal{O}_C)_a$-linear.

Transversality

By $\mathcal{O}_C$-linearity of the differential operators it suffices to look at the reduced operators.

- $(1 + \mathrm{I} \otimes J)\,\nabla\xi$ is surjective for generic J if φ_{red} is simple.
- $(1 + \mathrm{I} \otimes J)\,\not{D}\zeta^{1,0}$ can be shown to be surjective if M is of genus zero and N has positive holomorphic sectional curvature.
- A particularly good example are super J-holomorphic curves $\Phi\colon \mathbb{P}^{1|1}_{\mathbb{C}} \to \mathbb{P}^{n}_{\mathbb{C}}$.

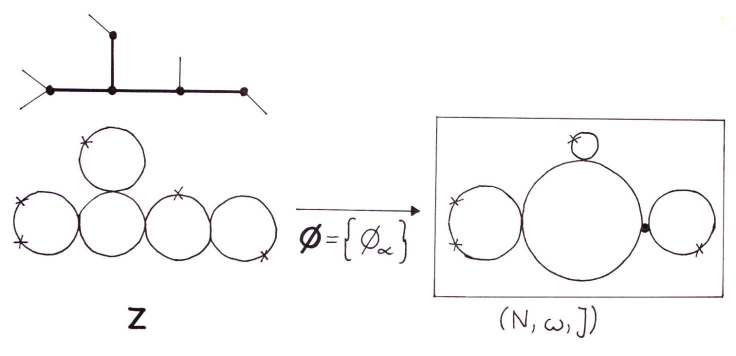
Geometry of the moduli space

- Suppose that the target N is a Kähler manifold and the domain super Riemann surface has vanishing gravitino, and the moduli space $\mathcal{M}(A)$ exists. Then $\mathcal{M}(A) = \operatorname{Split} K$ where $K \to M(A)$ is the bundle over the moduli space of (non-super) J-holomorphic curves such that $K_\phi = \ker \not{D}^{1,0}$.
- In that case the moduli space carries an almost complex structure induced from N.

Super Stable Maps

Compactification via stable maps

- The moduli space $\mathcal{M}_0(A)$ of classical J-holomorphic curves is in general not compact because sequences of J-holomorphic spheres might “bubble”. That is they might converge to trees of J-holomorphic curves.
- Certain bubbles of a bubble tree might be constant. Precomposing on a constant bubble with Möbius transformations leads to different descriptions of the same bubble tree.



Hence one generalizes to trees of bubbles with marked points.
This leads to so called stable J-holomorphic curves.

$$\overline{M}_{0,k}(A) = \bigcup_{k\text{-marked trees } T} \bigcup_{A=\sum A_\alpha} M_{0,T}(\{A_\alpha\})$$

Superconformal automorphisms of $\mathbb{P}^{1|1}_{\mathbb{C}}$

Automorphisms of the super Riemann surface $\mathbb{P}^{1|1}_{\mathbb{C}}$ are of the form

$$l^{\#}z_1 = \frac{az_1 + b}{cz_1 + d} \pm \theta_1 \frac{\gamma z_1 + \delta}{(cz_1 + d)^2}$$

$$l^{\#}\theta_1 = \frac{\gamma z_1 + \delta}{cz_1 + d} \pm \theta_1 \frac{1}{cz_1 + d}$$

with $ad - bc - \gamma\delta = 1$.

- Any three B-points of $\mathbb{P}^{1|1}_{\mathbb{C}}$ can be mapped by a unique superconformal automorphism to 0 ($z_1 = 0$, $\theta_1 = 0$), 1_ϵ ($z_1 = 1$, $\theta_1 = \epsilon$) and ∞ ($\frac{1}{z_1} = 0$, $\theta_1 = 0$).
- Any superconformal automorphism mapping $0 \mapsto 0$, $1_\epsilon \mapsto 1_{\epsilon'}$ and $\infty \mapsto \infty$ implies that $\epsilon = \pm\epsilon'$ and the map is either the identity or reflection of the odd directions ($l^{\#}\theta = -\theta$).

Definition
Let T be a k marked tree, represented by vertices $T = \{\alpha, \beta, \dots\}$, the edge matrix $E_{\alpha\beta}$ and the markings $\{1, \dots, k\} \to T$. A nodal supercurve of genus zero over B, modeled on T is a tuple

$$\mathbf{z} = \left(\{z_{\alpha\beta}\}_{E_{\alpha\beta}}, \{z_i\}_{1 \le i \le k}\right)$$

consisting of B-points $z_{\alpha\beta} \colon B \to \mathbb{P}^{1|1}_{\mathbb{C}}$ and $z_i \colon B \to \mathbb{P}^{1|1}_{\mathbb{C}}$ such that for every $\alpha \in T$ the reduction of the points $z_{\alpha\beta}$ and z_i for $p(i) = \alpha$ are disjoint. The $z_{\alpha\beta}$ are called nodal points and z_i are marked points.

The nodal curve is stable if at every node the number of special points, that is nodal points and marked points, is at least three.

Moduli space of stable supercurves

The moduli space of stable supercurves of fixed tree type T is a superorbifold of dimension

$$2k - 6 - 2e | 2k - 4$$

and singularities of type $\mathbb{Z}_2^{e+1}$.

It can be realized as the quotient by automorphisms of an open subsupermanifold of

$$\left(\mathbb{P}_{\mathbb{C}}^{1|1}\right)^{2e+k}.$$

The hard part of the proof is the definition of superorbifold and the slice theorem for Riemannian superorbifolds.

Definition
A super stable map of genus zero over B and modeled on T is a tuple

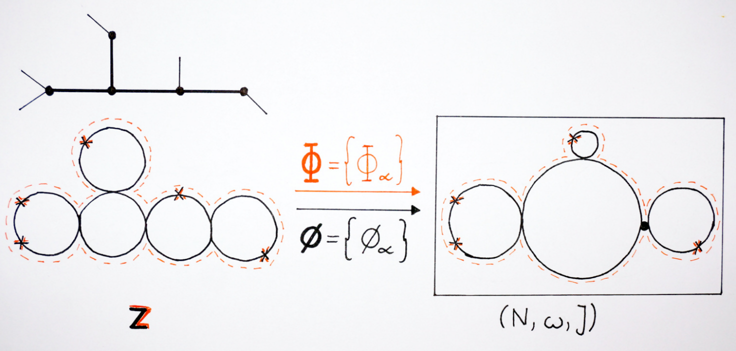
$$(\boldsymbol{z}, \Phi) = \left(\left(\{z_{\alpha\beta}\}_{E_{\alpha\beta}}, \{z_i\}_{1 \leq i \leq k} \right), \{\Phi_\alpha\}_{\alpha \in T} \right)$$

given by a nodal supercurve $\boldsymbol{z}$ and super J-holomorphic curves $\Phi_\alpha \colon \mathbb{P}^{1|1}_{\mathbb{C}} \times B \to N$ such that

- $\Phi_\alpha \circ z_{\alpha\beta} = \Phi_\beta \circ z_{\beta\alpha}$,
- the number of special points on nodes with constant Φ_α is at least three.

[9] Keßler, Sheshmani, and Yau (2020). *Super quantum cohomology I: Super stable maps of genus zero with Neveu-Schwarz punctures.*

Stable super J-holomorphic curves[10]



[10] Keßler, Sheshmani, and Yau (2020). *Super quantum cohomology I: Super stable maps of genus zero with Neveu-Schwarz punctures.*

Moduli space of super stable maps of fixed tree type

The moduli space $\mathcal{M}_{0,T}(\{A_\alpha\})$ of stable of stable super J-holomorphic curves of fixed tree type T and partition $\{A_\alpha\}_{\alpha \in T}$ of the homology class A is a superorbifold of dimension

$$2n + 2\langle A, c_1(TN)\rangle - 2e + 2k - 6 | 2\langle A, c_1(TN)\rangle + 2k - 4$$

and singularities of type $\mathbb{Z}_2^{e+1}$.

It can be realized as the quotient by automorphisms of a subsupermanifold of

$$\prod_{\alpha \in T} \mathcal{M}_0^*(A_\alpha) \times \left(\mathbb{P}_{\mathbb{C}}^{1|1}\right)^{2e+k}.$$

Moduli space of super stable maps

The moduli space of super stable maps with k marked points

$$\underline{\overline{\mathcal{M}}_{0,k}(A)}(B) = \bigcup_{k\text{-marked trees } T} \bigcup_{A=\sum A_\alpha} \underline{\mathcal{M}_{0,T}(\{A_\alpha\})}(B)$$

is not a superorbifold. Instead:

- We have proposed a generalization of Gromov topology.
- The reduced points form the compact space of classical stable maps.
- The reduction to fixed tree type T is a superorbifold.

Conclusions

- We have a definition of super J-holomorphic curve from a super Riemann surface to an almost Kähler manifold that mirrors and extends many of the properties of classical J-holomorphic curves.
- Under certain conditions on N we have constructed the moduli space of super J-holomorphic curves $\Phi\colon M \to N$ and its compactification in genus zero.
- We are working to understand the moduli space of super J-holomorphic curves better and search for super analogues of Gromov–Witten invariants.

References

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