Algebraic and combinatorial perspectives
in mathematical sciences

Topological recursion
from an algebraic perspective

Gaëtan Borot
HU Berlin
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I. Bottom-up : how 2d topology arises from algebra

II. Two examples : 2d TQFT and Virasoro constraints

III. Topological expansions in hermitian matrix models

IV. Top-down: from geometric to topological recursion

I

How 2d topology arises from algebra

I. How 2d topology arises from algebra — Airy structures

Let V be a vector space over $\mathbb{C}$

Choose a basis of linear coordinates $(x_i)_{i \in I}$

The Weyl algebra is the graded algebra of differential operators on V

$\mathcal{W}_V^\hbar = \mathbb{C}[\hbar]\langle x_i, \hbar\partial_{x_i} \; i \in I\rangle$ $\qquad \deg x_i = \deg \hbar\partial x_i = 1 \qquad \deg \hbar = 2$

An **Airy structure** is a linear map $L : V \to \mathcal{W}_V^\hbar$ such that

deg 1 condition : $L_i = \hbar\partial_{x_i} + O(2)$

ideal condition : $[L(V), L(V)] \subset \hbar\mathcal{W}_V^\hbar \cdot L(V)$

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I. How 2d topology arises from algebra — Airy structures

An **Airy structure** is a linear map $L : V \to \mathcal{W}_V^{\hbar}$ such that

deg 1 condition : $L_i = \hbar \partial_{x_i} + O(2)$ (uniqueness)

ideal condition : $[L(V), L(V)] \subset \hbar \mathcal{W}_V^{\hbar} \cdot L(V)$ (existence)

Theorem 1 (Kontsevich, Soibelman 17)

There exists a unique $F = \sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{g-1}}{n!} F_{g,n}$ with $F_{g,n} \in \mathrm{Sym}^n V^*$ such that

$\forall i \qquad L_i \cdot e^F = 0$ and $F_{0,1} = 0, F_{0,2} = 0$

e^F is called the partition function

I. How 2d topology arises from algebra — Partition function

Assume L_i has max. degree 2

$$L_i = \hbar\partial_{x_i} - \sum_{a,b}\left(\tfrac{1}{2}A^i_{a,b}x_a x_b + B^i_{a,b}x_a\hbar\partial_{x_b} + \tfrac{1}{2}C^i_{a,b}\hbar^2\partial_{x_a}\partial_{x_b}\right) - \hbar D^i$$

and decompose $F_{g,n} = \sum_{i_1,\ldots,i_n} F_{g,n}[i_1,\ldots,i_n]\, x_{i_1}\cdots x_{i_n}$

Let's compute $e^{-F}L_i \cdot e^F = 0$

$[\hbar^0\, \frac{x_j x_k}{2}]$ $\qquad F_{0,3}[i,j,k] - A^i_{j,k} = 0$

$[\hbar \cdot 1]$ $\qquad F_{1,1}[i] - D^i = 0$

$[\hbar^g\, \frac{x_{i_2}\cdots x_{i_n}}{(n-1)!}]$

$2g-2+n \geq 2$

$$F_{g,n}[i,i_2,\ldots,i_n] - \Bigg(\sum_a\sum_{m=2} B^i_{i_m,a}F_{g,n-1}[a,i_2,\ldots,\widehat{i_m},\ldots,i_n]$$
$$+\tfrac{1}{2}\sum_{a,b}C^i_{a,b}\Big(F_{g-1,n+1}[a,b,i_2,\ldots,i_n] + \sum_{\substack{I\sqcup I'=\{i_2,\ldots,i_n\}\\ h+h'=g}} F_{h,1+|J|}[a,I]\,F_{h',1+|J'|}[b,I']\Big)\Bigg) = 0$$

I. How 2d topology arises from algebra — Partition function

$$F_{g,n}[i, i_2, \ldots, i_n] - \Big(\sum_a \sum_{m=2} B^i_{i_m,a} F_{g,n-1}[a, i_2, \ldots, \widehat{i_m}, \ldots, i_n]$$
$$+ \tfrac{1}{2} \sum_{a,b} C^i_{a,b} \Big(F_{g-1,n+1}[a, b, i_2, \ldots, i_n] + \sum_{\substack{I \sqcup I' = \{i_2,\ldots,i_n\} \\ h+h'=g}} F_{h,1+|J|}[a, I]\, F_{h',1+|J'|}[b, I'] \Big)\Big) = 0$$

Take Σ smooth oriented surface, genus g, n labeled boundaries

P pair of pants with labeled boundaries

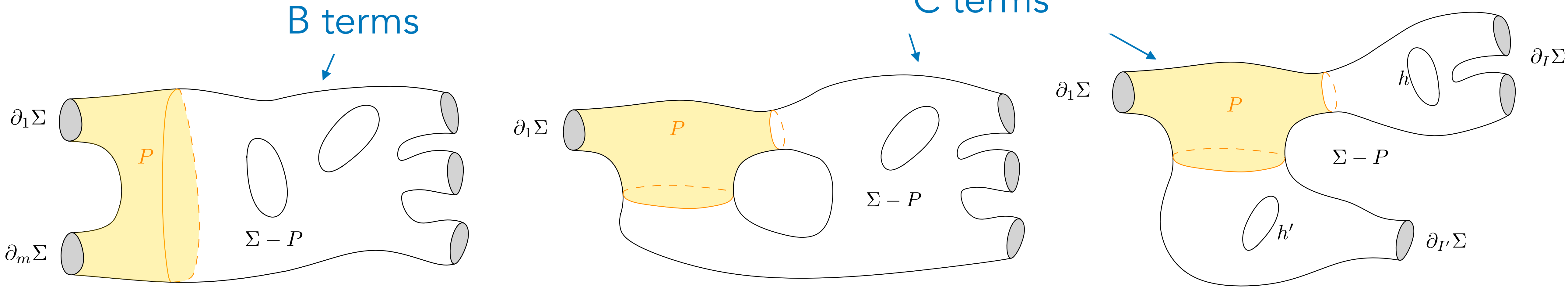
The terms in the bracket are in bijection with

$$\overline{\mathcal{P}}_\Sigma = \Big\{ P \hookrightarrow \Sigma \text{ such that } \partial_1 P = \partial_1 \Sigma \text{ and } \Sigma - P \text{ stable} \Big\} \Big/ \mathrm{Diff}^\partial_\Sigma$$

$$= \bigcup_{m=2}^{n} \Big\{ \ldots \text{ with } \partial_2 P = \partial_m \Sigma \Big\} \cup \Big\{ \ldots \text{ with } \partial_{2,3} P \subset \mathring{\Sigma} \Big\}$$

B terms

C terms



$\Longrightarrow$ $F_{g,n}$ uniquely determined by induction on $2g - 2 + n > 0$

I. How 2d topology arises from algebra — Partition function

- For higher order diff. op, we still get a recursion on $2g-2+n>0$

  but terms are now in bijection with $\left\{ \Sigma' \hookrightarrow \Sigma \text{ such that } \partial_1 \Sigma = \partial_1 \Sigma' \text{ and } \Sigma - \Sigma' \right\} \Big/ \mathrm{Diff}_{\Sigma}^{\partial}$

- The previous argument does not justify that $F_{g,n}[i, i_2, \ldots, i_n]$ is symmetric in $i \leftrightarrow i_k$

  This is a consequence of the ideal condition.

- In the quadratic case, this condition amounts to $[L_i, L_j] = \sum_a \hbar f_{i,j}^a L_a$

  i.e. $(L_i)_i$ forms a Lie algebra represented by atmost quadratic diff. op $f_{**}^* \in \mathbb{C}$

  $\rightsquigarrow$ $f_{i,j}^k = B_{j,k}^i - B_{i,k}^j$ and (overdetermined) quadratic relations for (A, B, C, D)

I. How 2d topology arises from algebra — Comments

- The ideal condition is hard to realise : exhibiting Airy structures in not obvious !

  The ones we know come from

  - cut and paste relations in 2d geometry
  - branched covers of complex curves (Eynard-Orantin theory)
  - conformal field theory (representation theory of VOAs)
  - Lie algebraic techniques (classification for semisimple Lie algebras Hadasz, Ruba (19))

- In many applications, the interpretation of g and n as genus and #boundaries of a surface is not artificial : $F_{g,n}$ "counts" such surfaces

  - maps (discretized surfaces), and so Feynman expansions of matrix integrals
  - branched covers
  - integrals over $\mathcal{M}_{g,n}$, $\mathcal{M}_{g,n}^{r\text{spin}}$, ...
  - Gromov-Witten invariants (integrals over $\mathcal{M}_{g,n}(X)$)

- Indirectly, applications to : knot theory, CFT, integrability, WKB expansions, etc.

II

Two examples : 2d TQFT, Virasoro constraints

II. Two examples — The 0th example

Take $V = \mathbb{C}$

$$L = \hbar\partial_x - \left(\tfrac{1}{2}x^2 + x\,\hbar\partial_x + \tfrac{\hbar^2}{2}\partial_x^2\right) - \hbar$$

Then $F_{g,n} = |\mathbb{G}_{g,n}| \in \mathbb{Z}[\tfrac{1}{2}]$

is the number of terms resulting from the unfolding
of the topological recursion, weighted by automorphisms

(~counts pairs of pants decomposition up to diffeo.)

In fact, the equation $L \cdot e^F = 0$ can be explicitly solved

$$e^F = \exp\left(\tfrac{1}{\hbar}\big(x - \tfrac{x^2}{2}\big)\right)\mathrm{Bi}\big(\tfrac{1-2x-\hbar}{(2\hbar)^{2/3}}\big)$$

where $\mathrm{Bi}(y) = y^{-1/4}\exp\big(-\tfrac{2}{3}y^{3/2}\big)\left(1 + \sum_{m\geq 1}\frac{6^m\Gamma(m+\frac{1}{6})\Gamma(m+\frac{5}{6})}{2\pi}\,\frac{y^{-3m/2}}{m!}\right)$

solves the Airy differential equation $\partial_y^2\mathrm{Bi}(y) = y\,\mathrm{Bi}(y)$

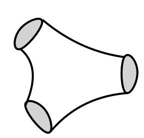
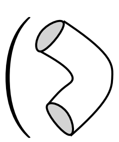
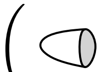
II. Two examples — 2d TQFT

Let Bord_2 be the monoidal category with

- objects : compact 1d smooth oriented manifolds
- morphisms : cobordisms
- monoidal structure : disjoint union

Let $\mathrm{Vect}_{\mathbb{C}}$ be the category of finite dim. vector spaces, monoidal structure $\otimes$

A **2d TQFT** is a monoidal functor $\mathcal{F} : \mathrm{Bord}_2 \to \mathrm{Vect}_{\mathbb{C}}$ (Atiyah)

It gives
- a vector space $\mathcal{F}(\mathbb{S}^1) = V$
- a product $\mathcal{F}\Big($  $\Big) = \mu \ : V^{\otimes 2} \to V$ commutative and associative
- a pairing $\mathcal{F}\Big($  $\Big) = b \ : V^{\otimes 2} \to \mathbb{C}$ symmetric and compatible : $b\big(\mu(a_1 \otimes a_2) \otimes a_3\big) = b\big(a_1 \otimes \mu(a_2 \otimes a_3)\big)$
- a unit $\mathcal{F}\Big($  $\Big) = \mathbf{1} \ : \mathbb{C} \to V$

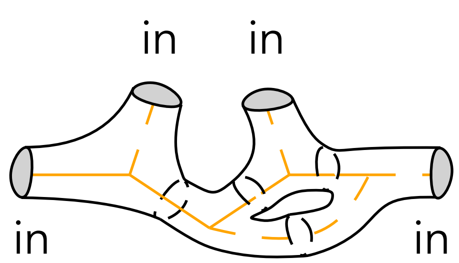
$\rightsquigarrow$ **Frobenius algebra**

A **2d TQFT** is a monoidal functor $\mathcal{F} : \mathrm{Bord}_2 \to \mathrm{Vect}_{\mathbb{C}}$ (Atiyah, 81)

Theorem (Abrams 96)

This a 1:1 correspondence between 2d TQFTs and Frobenius algebras

We can compute the TQFT functor from the Frobenius algebra
using some pair of pants decomposition of the cobordism

$\mathcal{F}\Big($ 

$\Big) = \left(\bigotimes_{\{c,c'\} \text{ glued}} b^*_{c,c'} \right) \circ \left(\bigotimes_{P = \text{pair of pants}} \mu^*_P \right) : V^{\otimes n} \to \mathbb{C}$

where $\mu^* \in (V^*)^{\otimes 3}$ and $b^* : V^* \otimes V^* \to \mathbb{C}$

By the Frobenius algebra axioms, the result is independent of the pair of pants
(hence matches the TQFT axioms)

II. Two examples — 2d TQFT

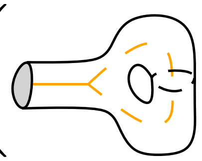
Lemma 2 (Andersen, B., Chekhov, Orantin 17)

Given a 2d TQFT, there is an Airy structure on $\mathcal{F}(\mathbb{S}^1) = V$

whose partition function generate $F_{g,n} = |\mathbb{G}_{g,n}| \cdot \mathcal{F}(\Sigma_{g,n\ \mathrm{in}})$

$$\begin{aligned} A &: V^{\otimes 3} \to \mathbb{C} \\ B &: V^{\otimes 2} \to V \\ C &: V \to V^{\otimes 2} \end{aligned}$$

represents the product when using $V \overset{b}{\simeq} V^*$

$D = \mathcal{F}\Big($  $\Big) : V \to \mathbb{C}$

Proved by comparison of TQFT rules with TR

The underlying Lie algebra is abelian because the product is symmetric

The interesting examples of Airy structures have infinite-dimensional V

Take $V = z\mathbb{C}[\![z^2]\!]$ with basis $e_k = \frac{z^{2k+1}}{2k+1}$, and define $e_k^* = \frac{(2k+1)\mathrm{d}z}{z^{2k+2}}$ $\quad k \in \mathbb{N}$

Take $\theta = \sum_{s \geq -1} \theta_s \, z^{2s} (\mathrm{d}z)^{-1}$

Introduce

$$\begin{cases} A^i_{j,k} = \operatorname*{Res}_{z=0} \big(e_i \cdot \mathrm{d}e_j \cdot \mathrm{d}e_k \cdot \theta\big) = \theta_{-1}\,\delta_{i,j,k,0} \\ B^i_{j,k} = \operatorname*{Res}_{z=0} \big(e_i \cdot \mathrm{d}e_j \cdot e^*_k \cdot \theta\big) = \frac{2k+1}{(2i+1)(2j+1)}\,(2j+1)\,\theta_{k-i-j} \\ C^i_{j,k} = \operatorname*{Res}_{z=0} \big(e_i \cdot e^*_j \cdot e^*_k \cdot \theta\big) = \frac{(2j+1)(2k+1)}{2i+1}\,\theta_{k+j+1-i} \\ D^i = \frac{\theta_0}{8}\,\delta_{i,0} + \frac{\theta_{-1}}{24}\,\delta_{i,1} \end{cases}$$

Lemma 3 (Kontsevich, Soibelman 17 ; Andersen, B., Chekhov, Orantin 17)

These (A,B,C,D) define a quadratic Airy structure based on a Lie algebra isomorphic to

$\mathrm{span}_{\mathbb{C}}(\mathcal{L}_i)_{i \geq s^*}$ with $[\mathcal{L}_i, \mathcal{L}_j] = (i-j)\mathcal{L}_{i+j}$ and $s^* = \min\{s \mid \theta_s \neq 0\}$

Intersection theory on $\overline{\mathcal{M}}_{g,n}$

$\mathcal{M}_{g,n} =$ moduli space of compact Riemann surfaces $\mathcal{C}$ of genus g with marked points $p_1, \ldots, p_n$

$\rightsquigarrow$ $\overline{\mathcal{M}}_{g,n}$ Deligne-Mumford compactification by allowing stable (nodal) curves

$$\psi_i = c_1(T^*_{p_i}\mathcal{C}) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

Witten's conjecture (Kontsevich + Dijkgraaf-Verlinde-Verlinde theorem, 91)

For $\theta = z^{-2}\mathrm{d}z$

$$F_{g,n}[k_1, \ldots, k_n] = \left(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{k_i}\right) \prod_{i=1}^{n} (2k_i - 1)!!$$

i.e. Virasoro constraints for ψ- class intersection

Wait until Part IV for a geometric explanation

Weil-Petersson volumes

$\mathcal{M}_{g,n}(L) =$ moduli space of bordered Riemann surfaces of genus g with n boundaries of lengths $L \in \mathbb{R}_+^n$

μ_{WP} Weil-Petersson volume form

Mirzakhani's recursion (07)

For $\theta = \dfrac{2\pi}{z\sin(2\pi z)\mathrm{d}z} = \sum_{s\geq -1} \zeta(2s+2)(2^{2s+3}-4)z^{2s}(\mathrm{d}z)^{-1}$

we have $\displaystyle\int_{\mathcal{M}_{g,n}(L)} \mathrm{d}\mu_{\mathrm{WP}} = \sum_{k_1,\ldots,k_n\geq 0} F_{g,n}[k_1,\ldots,k_n] \prod_{i=1}^{n} \frac{L_i^{2k_i}}{2k_i!}$

Wait until Part IV for a geometric explanation (due to Mirzakhani)

Operations on Airy structures

- $U = \exp\left(\frac{\hbar}{2}\sum_{a,b}\phi_{a,b}\partial_{x_a}\partial_{x_b}\right)$ acts by conjugation on $\mathcal{W}_V^{\hbar}$

  This amounts to $x_i \to x_i + \sum_a \phi_{i,a}\,\hbar\partial_{x_a}$

  hence preserves the notion of Airy structure

  $\rightsquigarrow$ Lemma 3 still applies when $\mathrm{d}e_i^* \to \mathrm{d}e_i^* + \sum_{a\geq 0}\phi_{i,a}\,\mathrm{d}e_a$

- Direct sums of Airy structures are Airy structures

  $\rightsquigarrow$ Lemma 3 has a generalisation to $V = zV_0[\![z^2]\!]$

  where V_0 is a Frobenius algebra

  (coupling of the Virasoro example with the 2d TQFT example)

Back to general θ. Let us define the involution $\sigma(z) = -z$ and the multidifferentials

$$\omega_{0,1}(z) = -\frac{1}{\theta}$$

$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} + \sum_{a,b \geq 0} \phi_{a,b}\, \mathrm{d}e_a(z_1)\mathrm{d}e_b(z_2)$$

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n \geq 0} F_{g,n}[k_1, \ldots, k_n] \prod_{i=1}^{n} e^*_{k_i}(z_i) \qquad 2g - 2 + n > 0$$

For any $g,\ n$

- $\omega_{g,n}(z, z_2, \ldots, z_n) + \omega_{g,n}(\sigma(z), z_2, \ldots, z_n)$ holomorphic at $z = 0$

  by definition

- $\omega_{g-1,n+1}(z, \sigma(z), z_2, \ldots, z_n) + \sum_{\substack{I \sqcup I' = \{z_2,\ldots,z_n\} \\ h+h'=g}} \omega_{h,1+|Z|}(z, I)\omega_{h',1+|I'|}(\sigma(z), I') = O\big(z^{2s^*}(\mathrm{d}z)^2\big)$

  equivalent to $\sum_{i \geq s^*} \frac{(\mathrm{d}z)^2}{z^{2i+1}} \mathcal{L}_i \cdot e^F = 0$

$\rightsquigarrow$ **abstract loop equations** (B., Eynard, Orantin 13)

More generally, there is a notion of abstract loop equations associated to the data of

$\mathcal{S}$ smooth complex curve

x, y meromorphic function on $\mathcal{S}$ such that

$\mathrm{d}x$ has finitely many zeroes, that are simple and not zeroes of $\mathrm{d}y$

$\omega_{0,2}$ symmetric bidifferential on $\mathcal{S}^2$ double pole with coef. 1 on the diagonal

- Frobenius algebra $V_0 = \bigoplus_{\mathrm{d}x(\alpha)=0} \mathbb{C}.e^{\alpha}$ orthonormal and $\mu(e^{\alpha} \otimes e^{\beta}) = \delta_{\alpha,\beta} e^{\alpha}$
- $\omega_{0,1} = y\mathrm{d}x$
- Locally near α : $x = x(\alpha) + z^2 \rightsquigarrow$ local involution $\sigma_{\alpha}(z) = -z$

$$\forall g, n, \alpha$$

$$\omega_{g,n}(z, z_2, \ldots, z_n) + \omega_{g,n}(\sigma_{\alpha}(z), z_2, \ldots, z_n) = O(\mathrm{d}z)$$

$$\omega_{g-1,n+1}(z, \sigma_{\alpha}(z), z_2, \ldots, z_n) + \sum_{\substack{I \sqcup I' = \{z_2,\ldots,z_n\} \\ h+h'=g}} \omega_{h,1+|I|}(z, I)\omega_{h',1+|I'|}(\sigma_{\alpha}(z), I') = O\big(y(z)(\mathrm{d}z)^2\big)$$

II. Two examples — Abstract loop equations

$$\forall g, n, \alpha$$

$$\omega_{g,n}(z, z_2, \ldots, z_n) + \omega_{g,n}(\sigma_\alpha(z), z_2, \ldots, z_n) = O(\mathrm{d}z)$$

$$\omega_{g-1,n+1}(z, \sigma_\alpha(z), z_2, \ldots, z_n) + \sum_{\substack{I \sqcup I' = \{z_2, \ldots, z_n\} \\ h+h'=g}} \omega_{h,1+|Z|}(z, I)\omega_{h',1+|I'|}(\sigma_\alpha(z), I') = O\big(y(z)(\mathrm{d}z)^2\big)$$

- Their set of solutions can be completely described (B., Shadrin, 15)

- There is a unique solution such that

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_\alpha \operatorname*{Res}_{z=\alpha} \left(\int_\alpha^z \omega_{0,2}(\cdot, z_1) \right) \omega_{g,n}(z, z_2, \ldots, z_n)$$

and it encodes the partition function of an Airy structure on $V = zV_0[\![z^2]\!]$

In particular this justifies existence and symmetry of the solution

- The ideal (here Lie) condition can be checked by direct computation

  but this looks ad hoc !

  There are two conceptual ways to find this Airy structure

  - it can be obtained from free field representation of the Virasoro algebra at c = 1

  - historically, Eynard-Orantin theory preexisted

- Other (higher order) Airy structures can be found from the free field rep. VOAs

  $W(\mathfrak{gl}_r)$ (Milanov 16 ; B., Bouchard, Chidambaram, Creutzig, Noshchenko 18 ; B., Kramer, Schüler 20)

  correspond to higher zeroes of $\mathrm{d}x$ and $\mathcal{S}$ possibly singular

  super-Virasoro (Bouchard, Ciosmak, Hadasz, Osuga, Ruba, Sulkowski 19)

  correspond to $\mathcal{S}$ = super-Riemann surface

III

Topological expansions in hermitian matrix models

Consider the probability measure on the space of hermitian matrices M of size N

$\mathrm{d}\mu(M) = \frac{\mathrm{d}M}{Z_N} e^{N\,\mathrm{Tr}\,V(M)}$ V : polynomial going to $-\infty$ at infinity

Define the correlators $W_n(x_1,\ldots,x_n) = \mathrm{Cumulant}_\mu\left(\mathrm{Tr}\,\frac{1}{x_1 - M},\ldots,\mathrm{Tr}\,\frac{1}{x_n - M}\right)$

By integration by parts, one can prove $\mu\Big[\big(\mathrm{Tr}\,\frac{1}{x-M}\big)^2 - \mathrm{Tr}\,\frac{N\,V'(M)}{x-M}\Big] = 0$

or equivalently $W_2(x,x) + W_1(x)^2 - N\,V'(x)W_1(x) = -N\big[V'(x)W_1(x)\big]_+$

Likewise, for each $n \geq 1$ there is a quadratic functional relation for $W_{n+1},\ldots,W_1$

$\rightsquigarrow$ **Schwinger-Dyson equations**

(Mhaskar, Saff, Totik,Anderson-Guionnet-Zeitouni …)

- As $N \to \infty$, the (random) spectral measure of M converges to some deterministic λ (almost surely and in expectation)

$$\rightsquigarrow \quad W_{0,1}(x) = \lim_{N\to\infty} \mu\Big[\tfrac{1}{N}\,\mathrm{Tr}\,\tfrac{1}{x-M}\Big] = \int \frac{\mathrm{d}\lambda(\xi)}{x-\xi}$$ exists, holomorphic in $x \in \mathbb{C} \setminus \mathrm{supp}\,\lambda$

(Tutte 60s, Brezin-Itzykson-Parisi-Zuber 81, …)

$\rightsquigarrow$ we have a spectral curve $\mathcal{S}\;:\;P(x,y) = y^2 - V'(x)y + \mathrm{Pol}(x) = 0$

on which $W_{0,1}(x)$ continues analytically to a meromorphic function

- Exploiting the Schwinger-Dyson equations and large deviation theory one can prove the existence of an asymptotic expansion $W_n \sim \sum_{g\geq 0} N^{2-2g-n}\,W_{g,n}$ when $\mathrm{supp}\,\lambda = [a,b]$ (t'Hooft 74, BIPZ 81, Pastur-Shcherbina 01, B. Guionnet 11)

- Then, $\omega_{g,n}(x_1,\ldots,x_n) = \left(W_{g,n}(x_1,\ldots,x_n) + \frac{\delta_{g,0}\delta_{n,2}}{(x_1-x_2)^2}\right)\prod_{i=1}^{n}\mathrm{d}x_i$ continues analytically to a meromorphic multidifferential on $\mathcal{S}^n$ (Eynard 04) with poles at $\mathrm{d}x_i = 0$ only (for $2g-2+n>0$)

- Inserting $W_n \sim \sum_{g \geq 0} N^{2-2g-n}\, W_{g,n}$ in the Schwinger-Dyson equations

  and *using analytic continuation implies* abstract loop equations for $(\omega_{g,n})_{g,n}$

  (B., Eynard, Orantin 13)

  Schwinger-Dyson equations themselves

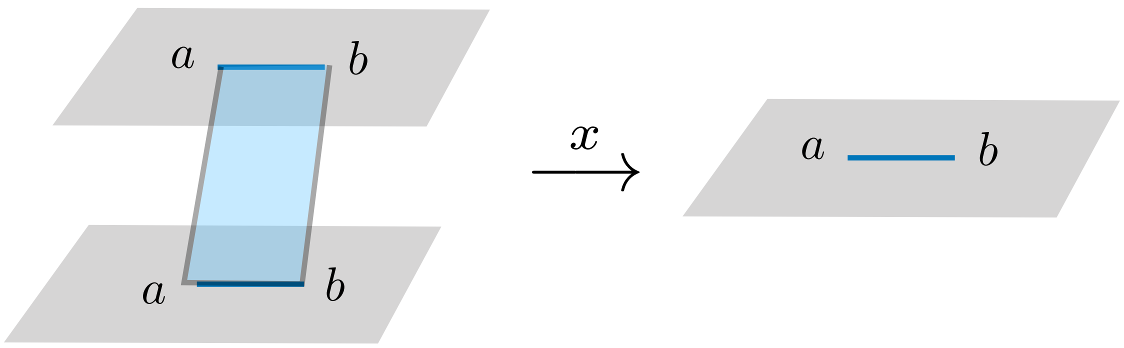
  (information near $x_i = \infty$, degree 1 condition fails)

  are not Airy structure constraints/abstract loop equations

  (information near $\mathrm{d}x = 0$)

- The assumption $\mathrm{supp}\,\lambda = [a,b]$ implies $\mathcal{S} \simeq \mathbb{P}^1$

  hence automatically $\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\alpha} \mathop{\mathrm{Res}}_{z=\alpha} \left(\int_{\alpha}^{z} \omega_{0,2}(\cdot, z_1) \right) \omega_{g,n}(z, z_2, \ldots, z_n)$ (Cauchy formula)

$\mathcal{S} =$ 

$\Longrightarrow$ $\omega_{g,n}$ computed

by topological recursion

(Eynard 05)

The same strategy applies to many other random hermitian matrix models

$$\mathrm{d}\mu(M) = \frac{\mathrm{d}M}{Z_N} \exp\Bigg(\sum_{\substack{p \geq 1 \\ m_1,\ldots,m_p \geq 1}} N^{2-p}\, t^{(p)}_{m_1,\ldots,m_p} \prod_{l=1}^{p} \mathrm{Tr}\, M^{m_l} \Bigg)$$

- existence of asymptotic expansions $W_n \sim \sum_{g \geq 0} N^{2-2g-n}\, W_{g,n}$ (B., Guionnet, Kozlowski, 15)
- SD implies abstract loop equations (B., Eynard, Orantin 13, B. 14)
- If $t^{(p)} = 0$ for all $p \geq 3$, the projection property holds and we have TR

  Otherwise, it does not and other solutions appear : **blobbed TR** (B. Shadrin 15)

- Blobbed TR appears in random colored tensor models (Eynard, Dartois, Bonzom, ...) and random spectral triples models (Azarfar's thesis, ...)

V

From geometric to topological recursion

We would like to lift TR to a natural construction associated to surfaces

Let Surf be the category with

- objects : compact smooth oriented stable surfaces with labeled boundaries
- morphisms : isotopy classes of orientation- and label-preserving diffeo.

Let $\mathcal{V}$ be a category of topological vector spaces

Assume we have a functor $E : \mathrm{Surf} \to \mathcal{V}$

An E-valued functorial assignment is the data of $\Omega_\Sigma \in E(\Sigma)$ for all objects Σ

such that, for any $f : \Sigma \to \Sigma'$ we have $E(f)(\Omega_\Sigma) = \Omega_{\Sigma'}$

In particular Ω_Σ is $\mathrm{Mod}_\Sigma^\partial := \mathrm{Diff}_\Sigma^\partial/(\mathrm{Diff}_\Sigma^\partial)_0$ -invariant

Geometric recursion constructs such functorial assignments by induction on $-\chi_\Sigma$

(Andersen, B., Orantin, 17)

- Teichmüller space

$$\mathcal{T}_\Sigma = \left\{ \begin{array}{l} \text{hyperbolic metrics on } \Sigma \\ \text{such that } \partial\Sigma \text{ is geodesic} \end{array} \right\} \Big/ (\mathrm{Diff}^{\partial}_{\Sigma})_0$$

Moduli space

$$\mathcal{M}_\Sigma = \mathcal{T}_\Sigma / \mathrm{Mod}^{\partial}_{\Sigma}$$

- $E(\Sigma) = \mathcal{C}^0(\mathcal{T}_\Sigma)$ with topology of convergence on all compacts

$\rightsquigarrow$ E-valued functorial assignments give continuous functions on the moduli space

- Let us look at

$$\mathcal{P}_\Sigma = \left(\bigcup_{m=2}^{n} \mathcal{P}^m_\Sigma \right) \cup \mathcal{P}^{\emptyset}_\Sigma$$

$$\mathcal{P}^{\emptyset}_\Sigma = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{l} \partial_1 P = \partial_1 \Sigma \\ \partial_2 P = \partial_m \Sigma \end{array} \right\}$$

$$\mathcal{P}^{m}_\Sigma = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{l} \partial_1 P = \partial_1 \Sigma \\ \partial_{2,3} P \subset \mathring{\Sigma} \end{array} \right\}$$

Its orbit set $\overline{\mathcal{P}}_\Sigma = \mathcal{P}_\Sigma / \mathrm{Mod}^{\partial}_\Sigma$ is finite and corresponds to the terms in TR

$P =$ pair of pants and note that $\mathcal{T}_P \cong \mathbb{R}_+^3$ (boundary lengths)
$T =$ torus with 1 boundary

Initial data $A, B, C \in \mathcal{C}^0(\mathbb{R}_+^3) \qquad D \in \mathcal{C}^0(\mathcal{T}_T)^{\mathrm{SL}_2(\mathbb{Z})}$

with A, C symmetric in their last 2 variables

GR construction

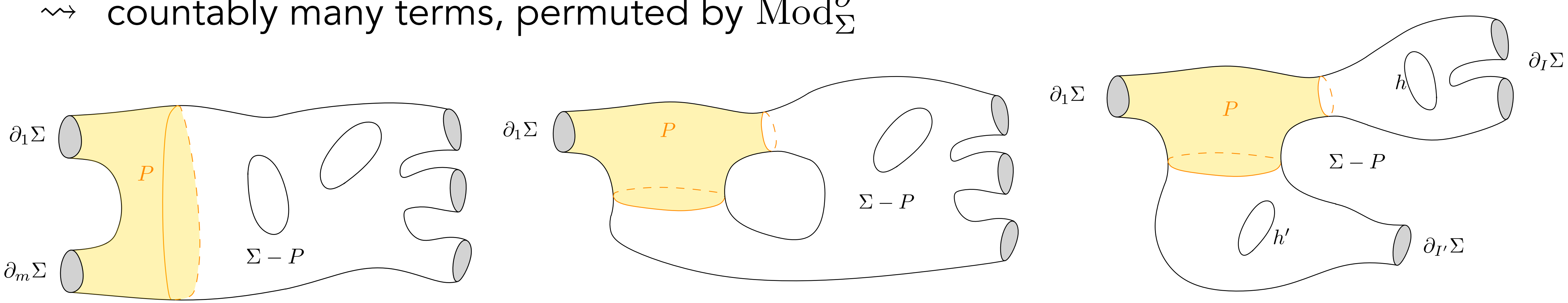
$\chi = -1$ $\qquad \Omega_P = A\big(\vec{\ell}(\partial P)\big)$ and $\Omega_T = D$

Disconnected $\qquad \Omega_{\Sigma_1 \sqcup \cdots \sqcup \Sigma_k}(\sigma_1, \ldots, \sigma_k) = \prod_{i=1}^{k} \Omega_{\Sigma_i}(\sigma_i)$

$\chi \leq -2$ $\qquad \Omega_\Sigma(\sigma) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^m} B\big(\vec{\ell}_\sigma(\partial P)\big)\,\Omega_{\Sigma-P}(\sigma|_{\Sigma-P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_\Sigma^\emptyset} C\big(\vec{\ell}_\sigma(\partial P)\big)\,\Omega_{\Sigma-P}(\sigma|_{\Sigma-P})$

by induction

$\rightsquigarrow$ countably many terms, permuted by $\mathrm{Mod}_\Sigma^\partial$



$\chi = -1$ $\quad\quad$ $\Omega_P = A\big(\vec{\ell}(\partial P)\big)$ and $\Omega_T = D$

Disconnected $\quad\quad$ $\Omega_{\Sigma_1 \sqcup \cdots \sqcup \Sigma_k}(\sigma_1, \ldots, \sigma_k) = \prod_{i=1}^{k} \Omega_{\Sigma_i}(\sigma_i)$

$\chi \leq -2$

by induction

$$\Omega_\Sigma(\sigma) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^m} B\big(\vec{\ell}_\sigma(\partial P)\big)\, \Omega_{\Sigma - P}(\sigma|_{\Sigma - P}) + \tfrac{1}{2} \sum_{[P] \in \mathcal{P}_\Sigma^\emptyset} C\big(\vec{\ell}_\sigma(\partial P)\big)\, \Omega_{\Sigma - P}(\sigma|_{\Sigma - P})$$

Theorem 4 (Andersen, B., Orantin, 17)

If A,B,C,D satisfy some decay conditions, then

- Ω_Σ is a well-defined functorial assignment (absolute convergence)
- $V\Omega_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \Omega_{\Sigma_{g,n}} \mathrm{d}\mu_{\mathrm{WP}}$ is a well-defined continuous function of $L \in \mathbb{R}_+^n$

and it safisfies topological recursion in the form :

$$V\Omega_{g,n}(L_1, \ldots, L_n) = \sum_{m=2}^{n} \int_{\mathbb{R}_+} \mathrm{d}\ell\, \ell\, B(L_1, L_m, \ell)\, V\Omega_{g,n-1}(\ell, L_2, \ldots, \widehat{L_m}, \ldots, L_n)$$

$$+ \tfrac{1}{2} \int_{\mathbb{R}_+^2} \mathrm{d}\ell \mathrm{d}\ell'\, \ell\ell'\, C(L_1, \ell, \ell') \Big(V\Omega_{g-1,n+1}(\ell, \ell', L_2, \ldots, L_n) + \sum_{\substack{I \sqcup I' = \{L_2, \ldots, L_n\} \\ h + h' = g}} V\Omega_{h,1+|I|}(\ell, I)\, V\Omega_{h',1+|I'|}(\ell', I') \Big)$$

with base cases $V\Omega_{0,3}(L_1, L_2, L_3) = A(L_1, L_2, L_3)$ and $V\Omega_{1,1}(L) = \int_{\mathcal{M}_{1,1}(L)} D\, \mathrm{d}\mu_{\mathrm{WP}}$

Take
$$\begin{cases} A_{\mathrm{M}}(L_1,L_2,L_3) = 1 \\ B_{\mathrm{M}}(L_1,L_2,\ell) = \dfrac{1}{2L_1}\big(F(L_1+L_2-\ell)+F(L_1-L_2-\ell)-F(-L_1+L_2-\ell)-F(-L_1-L_2-\ell)\big) \\ C_{\mathrm{M}}(L_1,\ell,\ell') = \dfrac{1}{L_1}\big(F(L_1-\ell-\ell')-F(-L_1-\ell-\ell')\big) \qquad \text{with } F(x) = 2\,\ln(1+e^{x/2}) \\ D_{\mathrm{M},T}(\sigma) = \displaystyle\sum_{\substack{\gamma=\text{simple} \\ \text{closed curve}}} C_{\mathrm{M}}\big(\ell_\sigma(\partial T),\ell_\sigma(\gamma),\ell_\sigma(\gamma)\big) \end{cases}$$

Theorem 4 (Mirzakhani, 07)

$\Omega_\Sigma(\sigma) = 1$ for any Σ and $\sigma \in \mathcal{T}_\Sigma$

As a consequence, $\int_{\mathcal{M}_{g,n}(L)} \mathrm{d}\mu_{\mathrm{WP}}$ satisfies the topological recursion

In fact, the integral operators with kernels B and C preserve the space of even polynomials

$$A(L_1,L_2,L_3) = \sum_{i,j,k\geq 0} A^i_{j,k}\, e_i(L_1)e_j(L_2)e_k(L_3)$$

with the basis $e_i(L) = \dfrac{L^{2i}}{(2i)!}$

$$\int_{\mathbb{R}_+} \mathrm{d}\ell\,\ell\, B(L_1,L_2,\ell)\, e_k(\ell) = \sum_{i,j\geq 0} B^i_{j,k}\, e_i(L_1)e_j(L_2)$$

yields the Airy structure we've seen before …

$$\int_{\mathbb{R}_+^2} \mathrm{d}\ell \mathrm{d}\ell'\, \ell\ell'\, C(L_1,\ell,\ell')\, e_j(\ell)e_k(\ell') = \sum_{i\geq 0} C^i_{j,k}\, e_i(L_1)$$

$$V\Omega_{1,1}(L) = \sum_{i\geq 0} D^i\, e_i(L)$$

The same thing can be carried on the combinatorial Teichmüller space

$$\mathcal{T}_{\Sigma}^{\mathrm{comb}} = \left\{ \begin{array}{l} \text{isotopy class of proper embeddings of metric ribbon graphs} \\ \mathbb{G} \overset{f}{\hookrightarrow} \Sigma \quad \text{such that } \Sigma \text{ retracts onto } f(\mathbb{G}) \text{ and labels agree} \end{array} \right\}$$

In his proof of Witten's conjecture, Kontsevich constructed a volume form $\mathrm{d}\mu_{\mathrm{K}}$ on the combinatorial Teichmüller space $\mathcal{M}_{\Sigma}^{\mathrm{comb}} = \dfrac{\mathcal{T}_{\Sigma}^{\mathrm{comb}}}{\mathrm{Mod}_{\Sigma}^{\partial}} = \displaystyle\bigcup_{\substack{G \text{ ribbon graph} \\ \text{type } (g,n)}} \dfrac{\mathbb{R}_{+}^{E(G)}}{\mathrm{Aut}\, G}$

so that $\displaystyle\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} \mathrm{d}\mu_{\mathrm{K}} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} L_i^2\, \psi_i\right)$

and used matrix model techniques to conclude

There is an analogue of Mirzakhani's theorem in the combinatorial case

Its integration produces the Virasoro constraint/Airy structure for ψ -intersections

$\rightsquigarrow$ geometric proof of Witten's conjecture

(Andersen, B., Charbonnier, Giacchetto, Lewanski, Wheeler, to appear)

Thank you for your attention !

2d TQFT
(exact factorization)

geometric recursion
(non local factorization)

integration on moduli space

Airy structures

partition function
computed by topological recursion

Abstract loop equations

conformal field theory

$H^*(\overline{\mathcal{M}}_{g,n})$

intersection theory

spectral curves

Schwinger-Dyson equations

matrix models and the like

A. Giacchetto