Algebra, geometry and physics seminar

Geometry of the combinatorial Teichmüller space (second part)

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based on joint works with Andersen, Charbonnier, Delecroix, Giacchetto, Lewanski, Orantin and Wheeler

Σ will usually denote a smooth bordered surface
oriented, connected (unless specified), genus g
n labeled boundaries $\partial_1\Sigma, \ldots, \partial_n\Sigma$
stable : $2 - 2g - n < 0$

I. Mirzakhani-type recursions

II. From geometric to topological recursion

III. Asymptotic counts of multicurves

Recall from last week …

We have two Teichmüller spaces

$\mathcal{T}_{\Sigma}(L)$ | $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$

They coincide as topological spaces, but carry different geometry

$\mathcal{T}_{\Sigma}(L)$	$\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$
{hyperbolic metrics}/Diff_0	{isotopy classes of metric ribbon graphs} = subset of {measured foliations}
smooth manifold	PL manifold
hyperbolic length functions	*combinatorial length functions*
hyperbolic Fenchel-Nielsen	*combinatorial Fenchel-Nielsen*
Darboux coords. for ω_{WP}	Darboux coords. for ω_{K}
full image in $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$	image = $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \setminus Z$

I

Mirzakhani-type recursion

I.1 Mirzakhani-type recursions — Hyperbolic geometry

- Recall

$$\mathcal{P}^{i}_{\Sigma} = \bigcup_{j=1}^{n} \mathcal{P}^{i,j}_{\Sigma}$$

$$\mathcal{P}^{i,j}_{\Sigma} = \left\{ \begin{array}{l} \text{homotopy class of } \; P \hookrightarrow \Sigma \\ \text{such that } \; \Sigma - P \; \text{ stable} \end{array} \;\middle|\; \begin{array}{l} \partial_1 P = \partial_i \Sigma \\ \partial_2 P = \partial_j \Sigma \end{array} \right\}$$

$$\mathcal{P}^{i,i}_{\Sigma} = \left\{ \begin{array}{l} \text{homotopy class of } \; P \hookrightarrow \Sigma \\ \text{such that } \; \Sigma - P \; \text{ stable} \end{array} \;\middle|\; \begin{array}{l} \partial_1 P = \partial_i \Sigma \\ \partial_{2,3} P \subset \mathring{\Sigma} \end{array} \right\}$$

- Define the functions

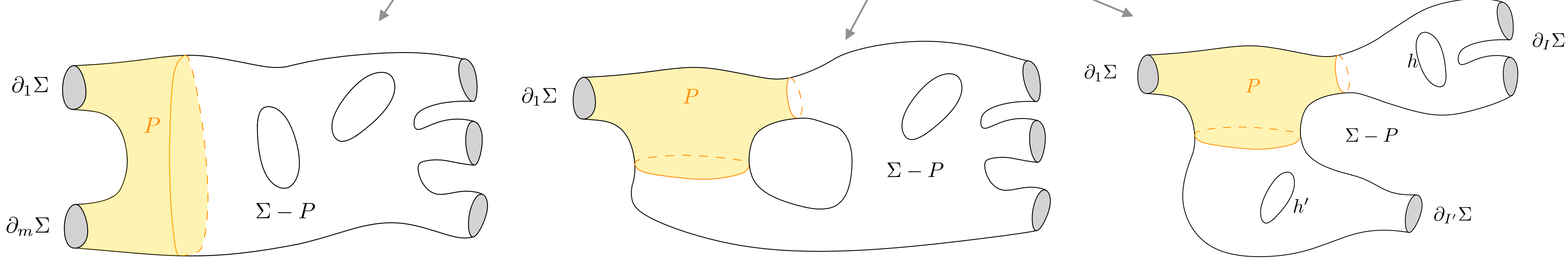
$$B_{\mathrm{M}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big(F(L_1 + L_2 - \ell) + F(L_1 - L_2 - \ell) - F(-L_1 + L_2 - \ell) - F(-L_1 - L_2 - \ell)\big)$$

$$C_{\mathrm{M}}(L_1, \ell, \ell') = \frac{1}{L_1}\big(F(L_1 - \ell - \ell') - F(-L_1 - \ell - \ell')\big) \qquad \text{with } \; F(x) = 2\,\ln(1 + e^{x/2})$$

Theorem (Mirzakhani, 07)

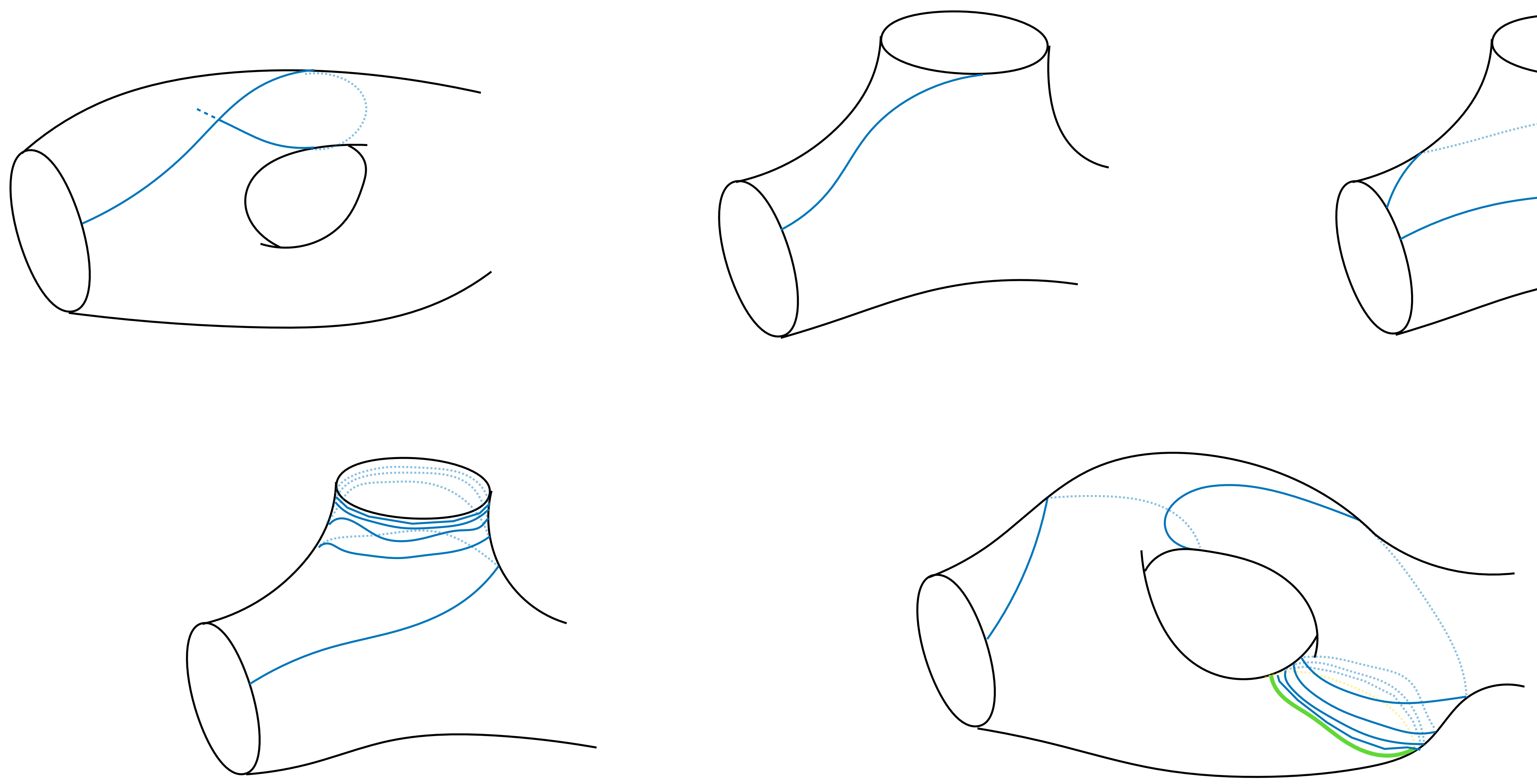
For $2g - 2 + n \geq 2$

$$\forall \sigma \in \mathcal{T}_{\Sigma} \qquad 1 = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}^{1,m}_{\Sigma}} B_{\mathrm{M}}\big(\vec{\ell}_{\sigma}(\partial P)\big) + \frac{1}{2} \sum_{[P] \in \mathcal{P}^{1,1}_{\Sigma}} C_{\mathrm{M}}\big(\vec{\ell}_{\sigma}(\partial P)\big)$$



Idea of the proof Let $\sigma \in \mathcal{T}_\Sigma$

$x \in \partial_1\Sigma \rightsquigarrow \gamma_x$ geodesic issuing from $x \perp \partial_1\Sigma$, stopped at first intersection point

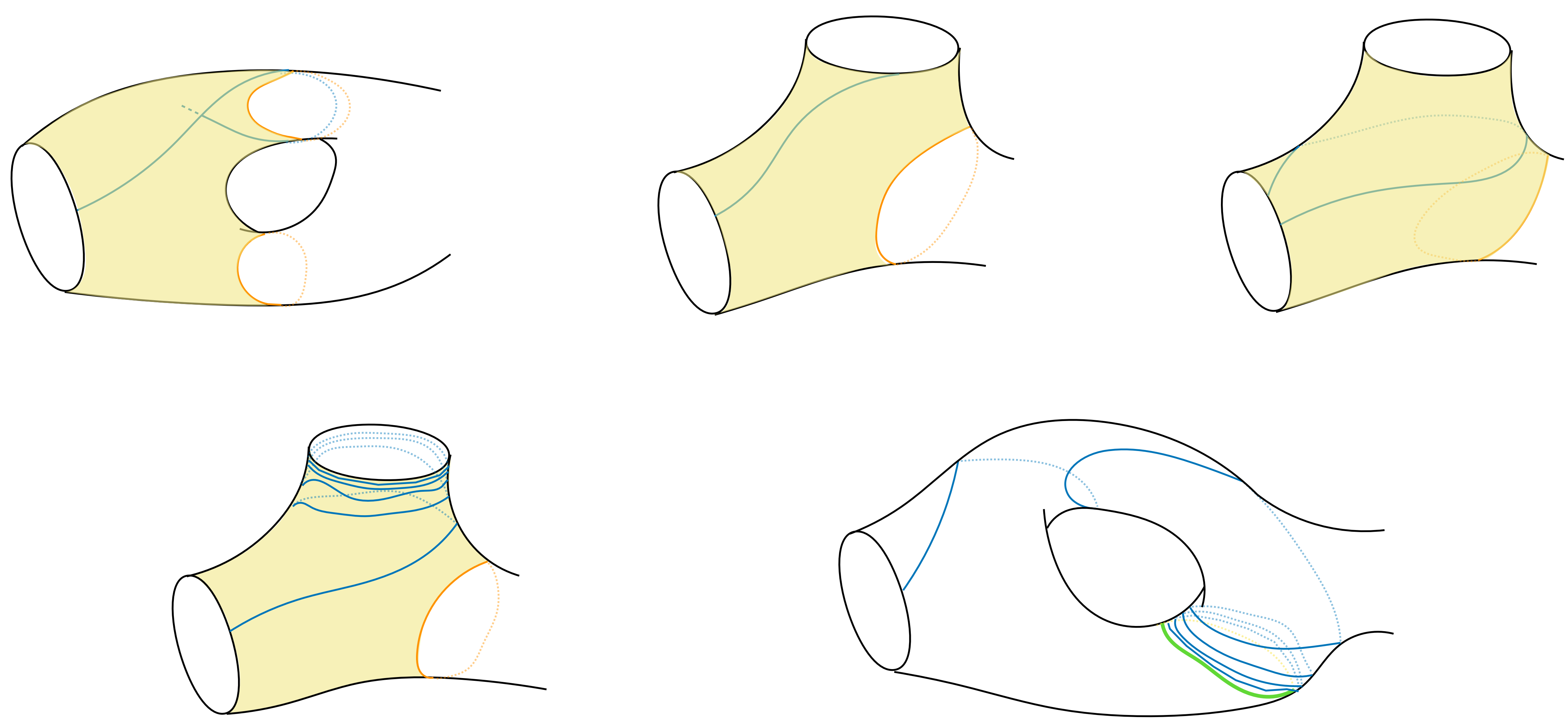


Idea of the proof Let $\sigma \in \mathcal{T}_\Sigma$

$x \in \partial_1 \Sigma \rightsquigarrow \gamma_x$ geodesic issuing from $x \perp \partial_1 \Sigma$, stopped at first intersection point

$\rightsquigarrow [P_x] \in \mathcal{P}^1_\Sigma$ determined by tubular neighboorhood of $\partial_1 \Sigma \cup \gamma_x$

when the geodesic does not accumulate on $\alpha \subset \mathring{\Sigma}$



(Birman-Series) The union of complete geodesics has Hausdorff dimension 1

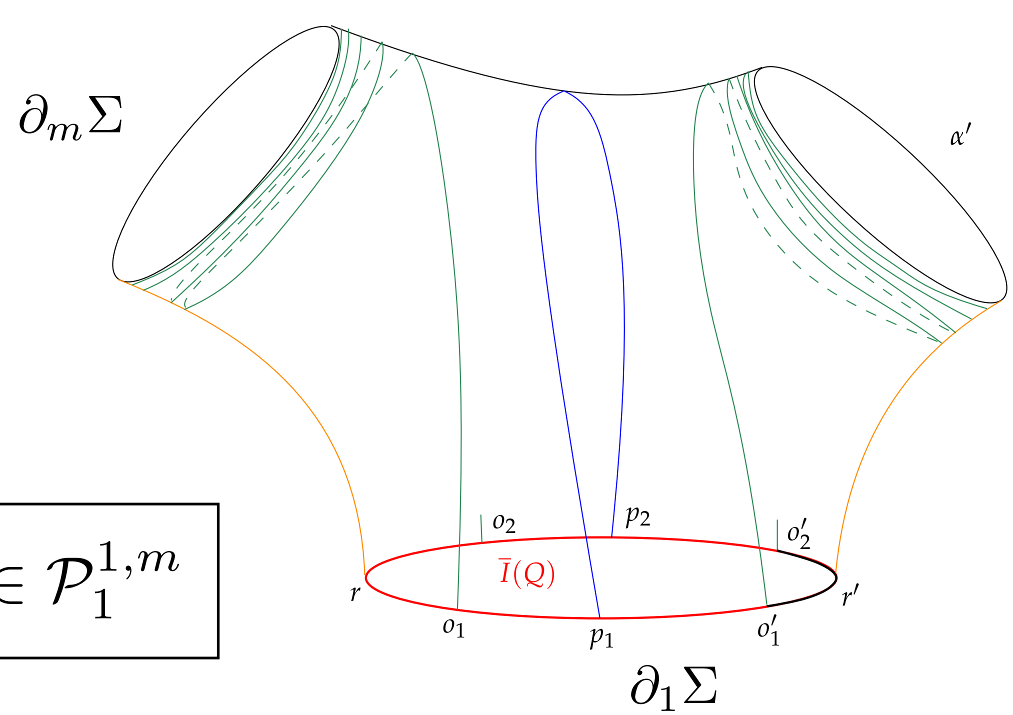
$\Longrightarrow \{x \in \partial_1 \Sigma \quad | \quad \gamma_x \text{ accumulates on } \alpha \subset \mathring{\Sigma}\}$ has Hausdorff dimension 0

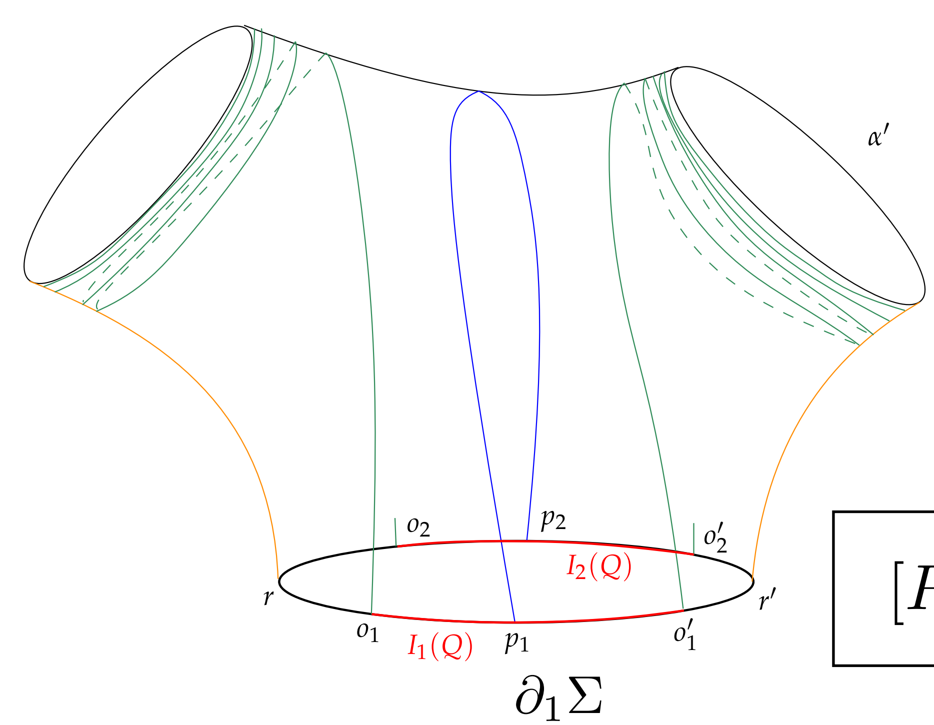
So we have an almost everywhere defined map $\partial_1 \Sigma \dashrightarrow \mathcal{P}^1_\Sigma$

$$1 = \frac{1}{\ell_\sigma(\partial_1 \Sigma)} \sum_{[P] \in \mathcal{P}^1_\Sigma} \ell_\sigma\big(\{x \in \partial_1 \Sigma \quad | \quad [P_x] = [P]\}\big)$$

Given $[P]$, one can identify the set of points $x \in \partial_1 \Sigma$ intrinsically and compute their measure by hyperbolic trigonometry

$[P] \in \mathcal{P}^{1,m}_1$





$[P] \in \mathcal{P}^{1,1}_1$

- Define the functions

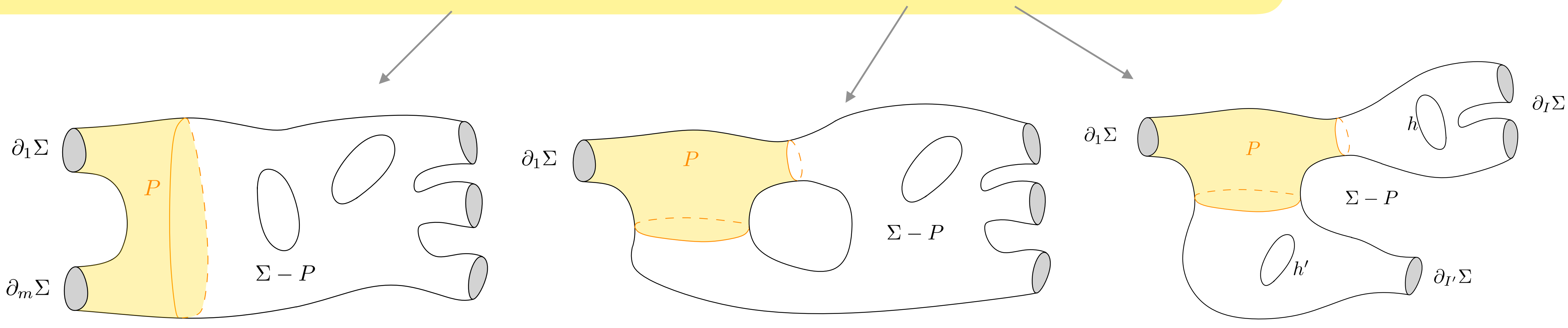
$$B_{\mathrm{K}}(L_1,L_2,\ell) = \frac{1}{2L_1}\big([L_1+L_2-\ell]_+ + [L_1-L_2-\ell]_+ - [-L_1+L_2-\ell]_+\big)$$

$$C_{\mathrm{K}}(L_1,\ell,\ell') = \frac{1}{L_1}[L_1-\ell-\ell']_+ \qquad\qquad [x]_+ = \max(x,0)$$

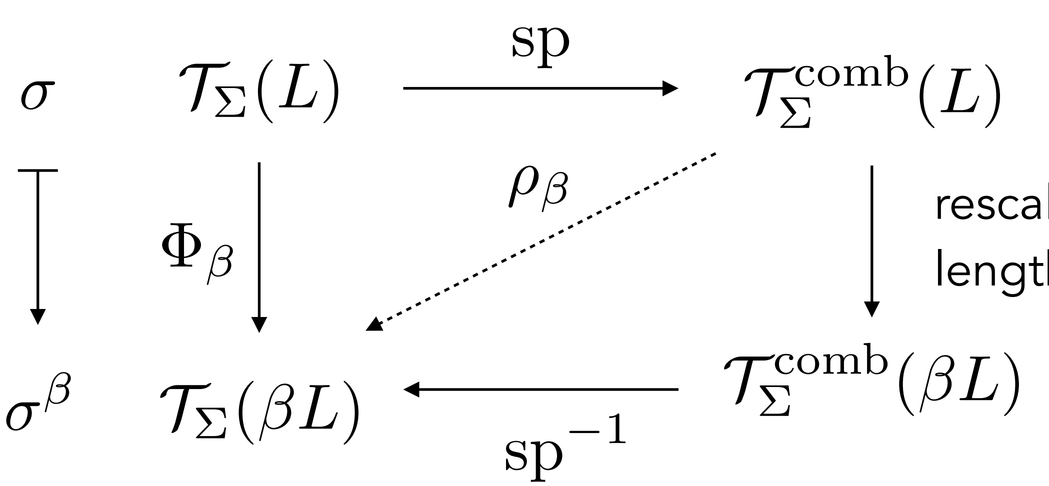
Theorem 1 **(ABCGLW, 20)**

For $2g-2+n \geq 2$

$$\forall \mathbb{G} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}} \qquad 1 = \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_{\Sigma}^{1,m}} B_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big) + \frac{1}{2}\sum_{[P]\in\mathcal{P}_{\Sigma}^{1,1}} C_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)$$



1st proof : flow the hyperbolic identity to the combinatorial one



$\beta^{-1}\,\ell_{\sigma^{\beta}}(\gamma) \to \ell_{\mathbb{G}}(\gamma)$
uniformly for γ
with $\mathbb{G} = \mathrm{sp}(\sigma)$

$$B_{\mathrm{M}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big(F(L_1 + L_2 - \ell) + F(L_1 - L_2 - \ell) - F(-L_1 + L_2 - \ell) - F(-L_1 - L_2 - \ell)\big)$$

$$C_{\mathrm{M}}(L_1, \ell, \ell') = \frac{1}{L_1}\big(F(L_1 - \ell - \ell') - F(-L_1 - \ell - \ell')\big)$$

with $F(x) = 2\,\ln(1 + e^{x/2})$

$$B_{\mathrm{K}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big([L_1 + L_2 - \ell]_+ + [L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+\big)$$

$$C_{\mathrm{K}}(L_1, \ell, \ell') = \frac{1}{L_1}[L_1 - \ell - \ell']_+$$

with $[x]_+ = \max(x, 0)$

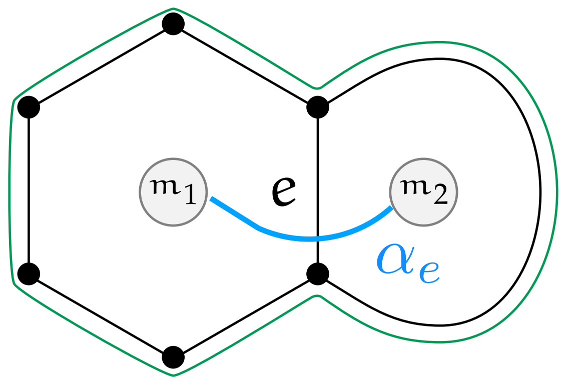
$$\frac{F(\beta x)}{\beta} = \frac{2}{\beta}\ln(1 + e^{\beta x/2}) \longrightarrow [x]_+ \qquad \Longrightarrow \begin{cases} B_{\mathrm{M}}\big(\vec{\ell}_{\sigma^{\beta}}(\partial P)\big) \longrightarrow B_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big) \\ C_{\mathrm{M}}\big(\vec{\ell}_{\sigma^{\beta}}(\partial P)\big) \longrightarrow C_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big) \end{cases}$$

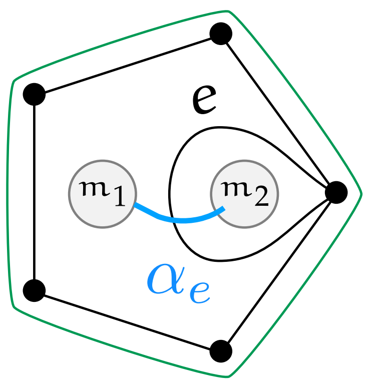
2nd proof (direct) Let $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}}$

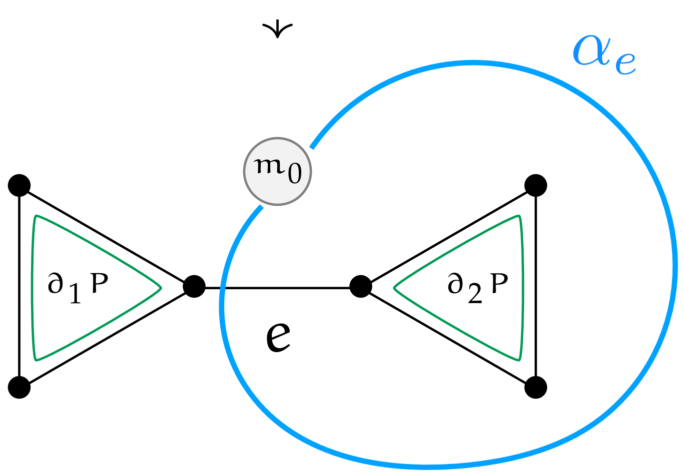
Write $$1 = \frac{1}{\ell_{\mathbb{G}}(\partial_1\Sigma)} \sum_{\substack{e = \text{edge} \\ \text{around } \partial_1\Sigma}} \ell_{\mathbb{G}}(e) = \frac{1}{\ell_{\mathbb{G}}(\partial_1\Sigma)} \sum_{\alpha \in \mathcal{A}_{\Sigma}^{1}} \ell_{\mathbb{G}}(\alpha) = \sum_{[P] \in \mathcal{P}_{\Sigma}^{1}} \sum_{\alpha \in Q^{-1}([P])} \ell_{\mathbb{G}}(\alpha)$$

where we recall $$\mathcal{A}_{\Sigma}^{1} = \left\{ \begin{array}{c} \text{homotopy class of} \\ \alpha : [0,1] \hookrightarrow \Sigma \end{array} \;\middle|\; \alpha(0) \in \partial_1\Sigma \right\} \xrightarrow{Q} \mathcal{P}_{\Sigma}^{1}$$

determined by the tubular neighborhood of $\partial_1\Sigma \cup \alpha$

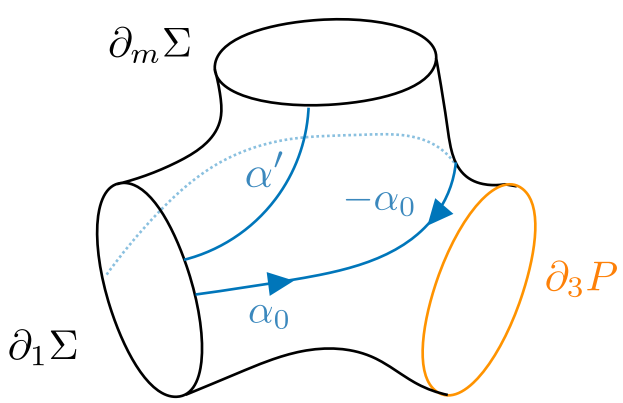


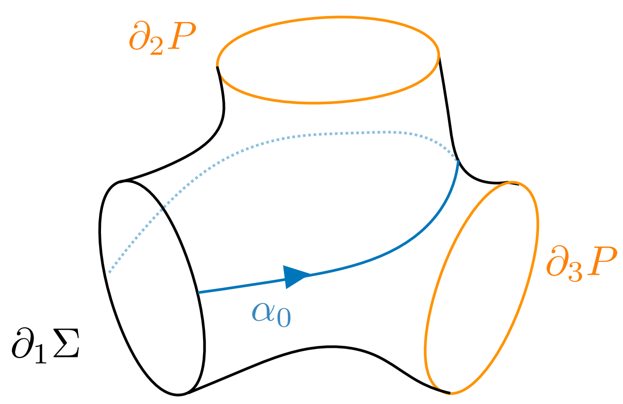




$$\mathcal{A}^1_\Sigma = \left\{ \begin{array}{c} \text{homotopy class of} \\ \alpha : [0,1] \hookrightarrow \Sigma \end{array} \,\middle|\, \alpha(0) \in \partial_1\Sigma \right\} \xrightarrow{Q} \mathcal{P}^1_\Sigma$$

has fibers $\quad Q^{-1}([P]) = \begin{cases} \{\alpha_0, -\alpha_0, \alpha'\} & \text{if } [P] \in \mathcal{P}^{1,m}_1 \; (m \neq 2) \\ \{\alpha_0\} & \text{if } [P] \in \mathcal{P}^{1,1}_\Sigma \end{cases}$





From last time we know that the length of an arc is given by

$$\ell_{\mathbb{G}}(\alpha) = \begin{cases} \ell_{\mathbb{G}}(\partial_1\Sigma)\Big[B_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial Q_\alpha)\big) - C_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial Q_\alpha)\big)\Big] & \text{if } Q_\alpha \in \mathcal{P}^{1,m}_\Sigma \; (m \neq 1) \\ \frac{1}{2}\ell_{\mathbb{G}}(\partial_1\Sigma)\, C_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial Q_\alpha)\big) & \text{if } Q_\alpha \in \mathcal{P}^{1,1}_\Sigma \end{cases}$$

$$\implies \frac{1}{\ell_{\mathbb{G}}(\partial_1\Sigma)} \sum_{\alpha \in Q^{-1}([P])} \ell_{\mathbb{G}}(\alpha) = \begin{cases} B_{\mathrm{K}}\big(\vec{\ell}(\partial P)\big) & \text{if } [P] \in \mathcal{P}^{1,m}_\Sigma \; (m \neq 1) \\ C_{\mathrm{K}}\big(\vec{\ell}(\partial P)\big) & \text{if } [P] \in \mathcal{P}^{1,1}_\Sigma \end{cases}$$

Let M_Σ (resp. M'_Σ) be the set of (primitive) multicurves on Σ

and $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that $\varphi(\ell) \underset{\ell \to \infty}{=} O(\ell^{-\infty})$

We consider multiplicative statistics of lengths of multicurves

- hyperbolic world : $\sigma \in \mathcal{T}_\Sigma$ $$\Omega_{\mathrm{M},\Sigma}[\varphi](\sigma) = \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_\sigma(\beta))$$
- combinatorial world : $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$ $$\Omega_{\mathrm{K},\Sigma}[\varphi](\mathbb{G}) = \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_{\mathbb{G}}(\beta))$$

Let us define

$$B[f](L_1, L_2, \ell) = B(L_1, L_2, \ell) + f(\ell)$$
$$C[f](L_1, \ell, \ell') = C(L_1, \ell, \ell') + B(L_1, \ell, \ell')f(\ell) + B(L_1, \ell', \ell)f(\ell') + f(\ell)f(\ell')$$

Theorem 2 (Andersen, B, Orantin 17)

For $2g - 2 + n \geq 2$ and any $\sigma \in \mathcal{T}_\Sigma$

$$\Omega_{\mathrm{M},\Sigma}[\varphi](\sigma) = \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^{1,m}} B_{\mathrm{M}}[\varphi]\big(\vec{\ell}_\sigma(\partial P)\big)\, \Omega_{\mathrm{M},\Sigma-P}[\varphi](\sigma_{|\Sigma-P}) + \frac{1}{2} \sum_{[P]\in\mathcal{P}_\Sigma^{1,1}} C_{\mathrm{M}}[\varphi]\big(\vec{\ell}_\sigma(\partial P)\big)\, \Omega_{\mathrm{M},\Sigma-P}[\varphi](\sigma_{|\Sigma-P})$$

Theorem 3 (ABCGLW 20)

For $2g - 2 + n \geq 2$ and any $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$

$$\Omega_{\mathrm{K},\Sigma}[\varphi](\mathbb{G}) = \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^{1,m}} B_{\mathrm{K}}[\varphi]\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\, \Omega_{\mathrm{K},\Sigma-P}[\varphi](\mathbb{G}_{|\Sigma-P}) + \frac{1}{2} \sum_{[P]\in\mathcal{P}_\Sigma^{1,1}} C_{\mathrm{K}}[\varphi]\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\, \Omega_{\mathrm{K},\Sigma-P}[\varphi](\mathbb{G}_{|\Sigma-P})$$

Idea of the proof same in hyperbolic or combinatorial setting

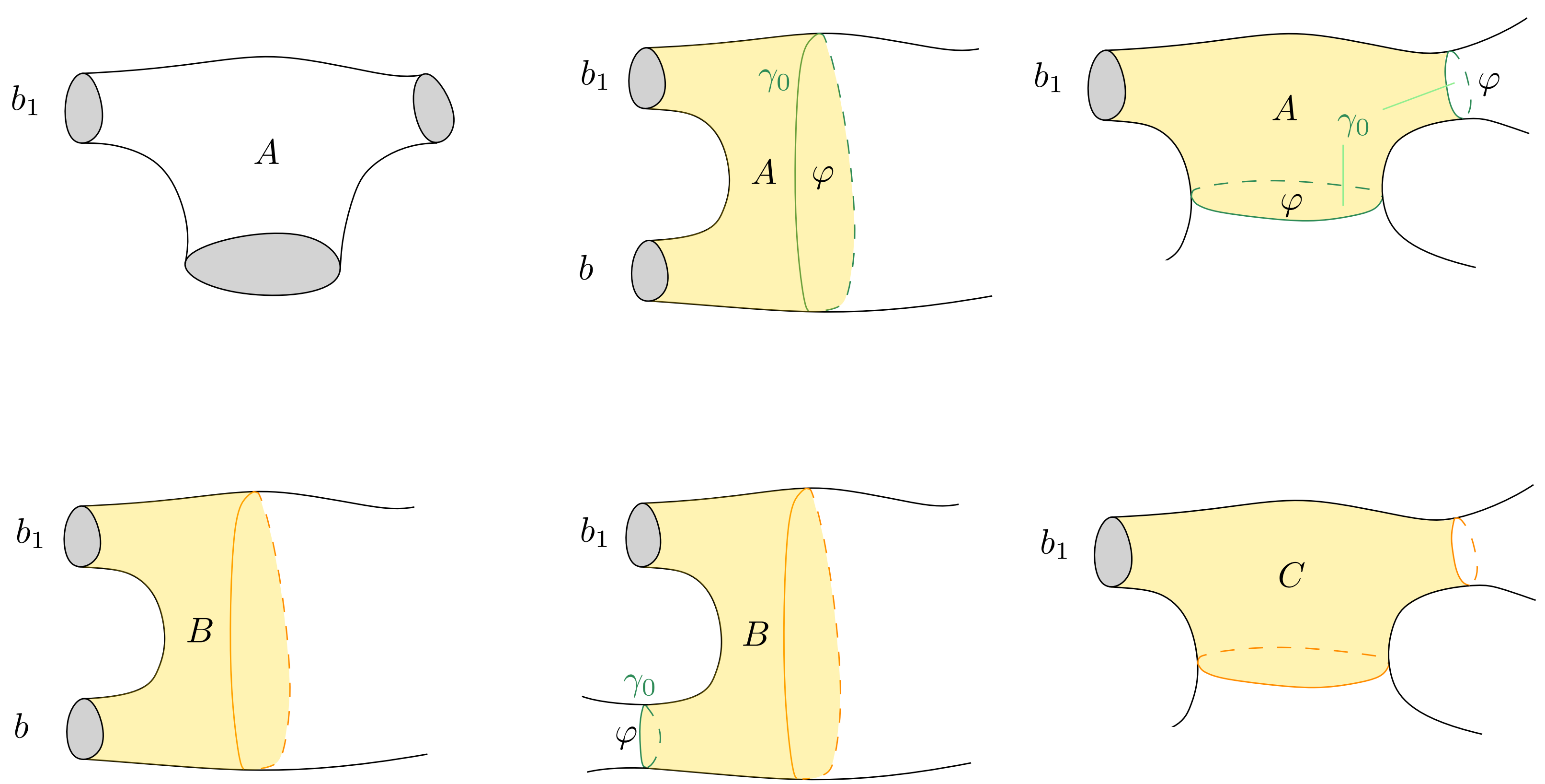
$$\Omega_{\mathrm{M}}[\varphi](\sigma) = \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_\sigma(\beta)) \cdot \mathbf{1}_{\Sigma-\gamma}(\sigma|_{\Sigma-\gamma})$$

use previous identity

$$= \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_\sigma(\gamma)) \sum_{[P] \in \mathcal{P}^1_{\Sigma-\gamma}} X_{\mathrm{M},P}(\sigma|_{\Sigma-P})$$

$$= \sum_{[P] \in \mathcal{P}^1_\Sigma} \sum_{\gamma \in M'_{\Sigma-P}} \cdots$$

and collect the weights



$A = \Omega_{0,3} \equiv 1$

II

From geometric recursion to topological recursion

II.1 From GR to TR — Hyperbolic

Geometric recursion produces, among others, $\mathrm{Mod}_{\Sigma}^{\partial}$-invariant functions on $\mathcal{T}_{\Sigma}$

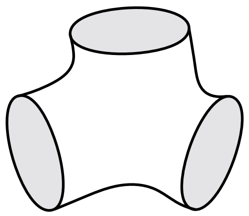
$\Omega_{\Sigma} \in C^0(\mathcal{T}_{\Sigma})^{\mathrm{Mod}_{\Sigma}^{\partial}}$ by induction on $2g - 2 + n > 0$

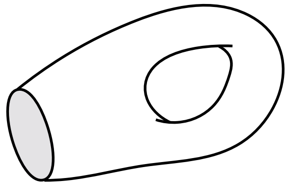
Initial data

- $\Omega_P, B, C \in C^0(\mathcal{T}_P)^{\mathrm{Mod}_P^{\partial}} \cong C^0(\mathbb{R}_+^3)$

  Ω_P, C symmetric in last two variables

- $\Omega_T \in C^0(\mathcal{T}_T)^{\mathrm{Mod}_T^{\partial}}$

$P =$ 

$T =$ 

Recursion scheme

For disconnected surfaces $\Omega_{\Sigma_1 \cup \cdots \cup \Sigma_k}(\sigma_1, \ldots, \sigma_k) = \prod_{i=1}^{k} \Omega_{\Sigma_i}(\sigma_i)$

For connected, $2g - 2 + n \geq 2$

$$\Omega_{\Sigma}(\sigma) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_{\Sigma}^{1,m}} B\big(\vec{\ell}_{\sigma}(\partial P)\big)\, \Omega_{\Sigma - P}(\sigma_{|\Sigma - P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_{\Sigma}^{1,1}} C\big(\vec{\ell}_{\sigma}(\partial P)\big)\, \Omega_{\Sigma - P}(\sigma_{|\Sigma - P})$$

$$\Omega_\Sigma(\sigma) = \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^{1,m}} B\big(\vec{\ell}_\sigma(\partial P)\big)\,\Omega_{\Sigma-P}(\sigma_{|\Sigma-P}) + \frac{1}{2}\sum_{[P]\in\mathcal{P}_\Sigma^{1,1}} C\big(\vec{\ell}_\sigma(\partial P)\big)\,\Omega_{\Sigma-P}(\sigma_{|\Sigma-P})$$

Theorem 4 (Andersen, B, Orantin 17)

Assume $\Omega_P, \Omega_T \in O(1)$

$$|B(L_1,L_2,\ell)| \in O\big(\ell^{-2+}\,(1+[\ell-L_1-L_2]_+)^{-\infty}\big)$$

$$|C(L_1,\ell,\ell')| \in O\big((\ell\,\ell')^{-2+}\,(1+[\ell+\ell'-L_1]_+)^{-\infty}\big)$$

Then, $\Omega_\Sigma \in C^0(\mathcal{T}_\Sigma)^{\mathrm{Mod}_\Sigma^\partial}$ and $V_{g,n}(L) = \int_{\mathcal{M}_\Sigma(L)} \Omega_\Sigma(\sigma)\,\mathrm{d}\mu_{\mathrm{WP}}(\sigma)$ are well-defined

and the integrals satisfy the recursion on $2g-2+n\geq 2$

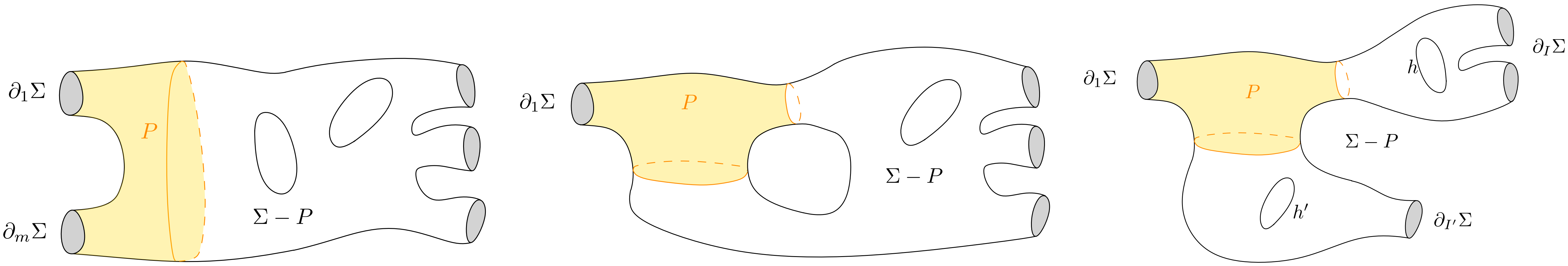
$$V\Omega_{g,n}(L_1,\ldots,L_n) = \sum_{m=2}^{n}\int_{\mathbb{R}_+}\mathrm{d}\ell\,\ell\,B(L_1,L_m,\ell)\,V\Omega_{g,n-1}(\ell,L_2,\ldots,\widehat{L_m},\ldots,L_n)$$

$$+\frac{1}{2}\int_{\mathbb{R}_+^2}\mathrm{d}\ell\,\mathrm{d}\ell'\,C(L_1,\ell,\ell')\Big(V\Omega_{g,n-1}(\ell,\ell',L_2,\ldots,L_n) + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{L_2,\ldots,L_n\}}} V\Omega_{h,1+|J|}(\ell,J)\,V\Omega_{h',1+|J'|}(\ell',J')\Big)$$

II.1 From GR to TR — Hyperbolic

$$V\Omega_{g,n}(L_1,\dots,L_n) = \sum_{m=2}^{n} \int_{\mathbb{R}_+} \mathrm{d}\ell\,\ell\, B(L_1,L_m,\ell)\, V\Omega_{g,n-1}(\ell,L_2,\dots,\widehat{L_m},\dots,L_n)$$

$$+\frac{1}{2}\int_{\mathbb{R}_+^2} \mathrm{d}\ell\,\mathrm{d}\ell'\, C(L_1,\ell,\ell')\Big(V\Omega_{g,n-1}(\ell,\ell',L_2,\dots,L_n) + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{L_2,\dots,L_n\}}} V\Omega_{h,1+|J|}(\ell,J)\, V\Omega_{h',1+|J'|}(\ell',J')\Big)$$



These terms are in bijection with the finite set $\mathcal{P}^1_\Sigma/\mathrm{Mod}^\partial_\Sigma$

$[[P]] \in \mathcal{P}^1_\Sigma/\mathrm{Mod}^\partial_\Sigma$ characterized by the topology of $\Sigma - P$

Idea of the proof

- $\left|\{\gamma \in S_\Sigma \quad | \quad \ell_\sigma(\gamma) \leq \beta\}\right| \leq C_{\mathrm{sys}(\sigma)}\, \beta^{6g-6+2n}$

  allows proving the sums are absolutely convergent on $\{\mathrm{sys}_\sigma \geq \epsilon\} \subset \mathcal{T}_\Sigma$

- Let $[P_0] \in \mathcal{P}^1_\Sigma$ $\qquad \mathrm{Mod}^\partial_\Sigma.[P_0] \cong \mathrm{Mod}^\partial_\Sigma/\mathrm{Stab}([P_0]) = \mathcal{O}$

  $\forall f \in \mathrm{Mod}^\partial_\Sigma \qquad \vec{\ell}_\sigma(f(\partial P_0)) = \vec{\ell}_{f^{-1}(\sigma)}(\partial P_0)$

  $\Omega_{\Sigma - f(P_0)}(\sigma_{|\Sigma - f(P_0)}) = \Omega_{\Sigma - P_0}\big(f(\sigma)_{|\Sigma - P_0}\big)$

$$\Longrightarrow \int_{\mathcal{M}_\Sigma(L)} \Bigg(\sum_{[P] \in \mathcal{O}} X\big(\vec{\ell}_\sigma(\partial P)\big)\, \Omega_{\Sigma - P}(\sigma_{|\Sigma - P})\Bigg) \mathrm{d}\mu_{\mathrm{WP}}(\sigma) = \int_{\mathcal{T}_\Sigma(L)/\mathrm{Stab}([P_0])} X\big(\vec{\ell}_\sigma(\partial P_0)\big)\, \Omega_{\Sigma - P_0}(\sigma_{|\Sigma - P_0})$$

- Take a seamed pair of pants decomposition of Σ containing P_0

$$\mathrm{d}\mu_{\mathrm{WP}} = \prod_i \mathrm{d}\ell_i\, \mathrm{d}\tau_i \prod_{\alpha \in \pi_0(\partial P \cap \mathring{\Sigma})} \mathrm{d}\ell_\alpha\, \mathrm{d}\tau_\alpha \qquad \text{(Wolpert formula)}$$

$$\mathcal{T}_\Sigma(L)/\mathrm{Stab}([P_0]) \simeq \bigcup_\ell \mathcal{M}_{\Sigma - P_0}((\ell_\alpha)_\alpha, L_{\Sigma - P_0}) \times \big\{(\ell_\alpha, \tau_\alpha)_\alpha\big\}/\tau_\alpha \to \tau_\alpha + \ell_\alpha$$

II.1 From GR to TR — Hyperbolic

Mirzakhani's theorem realizes the constant function 1 as an outcome of GR

$$\Omega_P = 1$$

$$B_{\mathrm{M}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big(F(L_1 + L_2 - \ell) + F(L_1 - L_2 - \ell) - F(-L_1 + L_2 - \ell) - F(-L_1 - L_2 - \ell)\big)$$

$$C_{\mathrm{M}}(L_1, \ell, \ell') = \frac{1}{L_1}\big(F(L_1 - \ell - \ell') - F(-L_1 - \ell - \ell')\big) \qquad \text{with} \quad F(x) = 2\,\ln(1 + e^{x/2})$$

$$\Omega_T(\sigma) = \sum_{\gamma \in \mathcal{S}_T} C_{\mathrm{M}}\big(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)\big) = 1$$

$\Longrightarrow$ topological recursion for the Weil-Petersson volumes (Mirzakhani, 07)

$$V^{\mathrm{WP}}_{g,n}(L) = \int_{\mathcal{M}_\Sigma(L)} \mathrm{d}\mu_{\mathrm{WP}}(\sigma)$$

The same can be done in the combinatorial setting

$$\Omega_{\Sigma}(\mathbb{G}) = \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_{\Sigma}^{1,m}} B\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\,\Omega_{\Sigma-P}(\mathbb{G}_{|\Sigma-P}) + \frac{1}{2}\sum_{[P]\in\mathcal{P}_{\Sigma}^{1,1}} C\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\,\Omega_{\Sigma-P}(\mathbb{G}_{|\Sigma-P})$$

Theorem 5 (ABCGLW, 20)

Assume $\Omega_P, \Omega_T \in O(1)$

$$|B(L_1, L_2, \ell)| \in O\big(\ell^{-2+}\,(1 + [\ell - L_1 - L_2]_+)^{-\infty}\big)$$

$$|C(L_1, \ell, \ell')| \in O\big((\ell\,\ell')^{-2+}\,(1 + [\ell + \ell' - L_1]_+)^{-\infty}\big)$$

Then, $\Omega_{\Sigma} \in C^0(\mathcal{T}_{\Sigma}^{\mathrm{comb}})^{\mathrm{Mod}_{\Sigma}^{\partial}}$ and $V\Omega_{g,n}(L) = \int_{\mathcal{M}_{\Sigma}^{\mathrm{comb}}(L)} \Omega_{\Sigma}(\mathbb{G})\,\mathrm{d}\mu_{\mathrm{K}}(\mathbb{G})$ are well-defined

and the integrals satisfy the recursion on $2g - 2 + n \geq 2$

$$V\Omega_{g,n}(L_1,\ldots,L_n) = \sum_{m=2}^{n} \int_{\mathbb{R}_+} \mathrm{d}\ell\,\ell\,B(L_1, L_m, \ell)\,V\Omega_{g,n-1}(\ell, L_2, \ldots, \widehat{L_m}, \ldots, L_n)$$

$$+\frac{1}{2}\int_{\mathbb{R}_+^2} \mathrm{d}\ell\,\mathrm{d}\ell'\,C(L_1,\ell,\ell')\Big(V\Omega_{g,n-1}(\ell,\ell',L_2,\ldots,L_n) + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{L_2,\ldots,L_n\}}} V\Omega_{h,1+|J|}(\ell, J)\,V\Omega_{h',1+|J'|}(\ell', J')\Big)$$

Particularities of the proof

- $\big|\{\gamma \in S_\Sigma \quad | \quad \ell_{\mathbb{G}}(\gamma) \leq \beta\}\big| \leq C_{\mathrm{sys}(\mathbb{G})}\, \beta^{6g-6+2n}$ also holds

- Again $\displaystyle\int_{\mathcal{M}_\Sigma^{\mathrm{comb}}(L)} \sum_{[P] \in \mathcal{O}} \cdots = \int_{\mathcal{T}_\Sigma^{\mathrm{comb}}(L)/\mathrm{Stab}([P_0])} \cdots$

- Combinatorial Wolpert formula (last week)

$$\mathrm{d}\mu_{\mathrm{K}} = \prod_i \mathrm{d}\ell_i\, \mathrm{d}\tau_i \prod_{\alpha \in \pi_0(\partial P \cap \mathring{\Sigma})} \mathrm{d}\ell_\alpha\, \mathrm{d}\tau_\alpha$$

$$\mathcal{T}_\Sigma^{\mathrm{comb}}(L)/\mathrm{Stab}([P_0]) \simeq \bigcup_\ell \mathcal{M}_{\Sigma - P_0}^{\mathrm{comb}}((\ell_\alpha)_\alpha, (L_i)_i) \times \big\{(\ell_\alpha, \tau_\alpha)_\alpha\big\}/\tau_\alpha \to \tau_\alpha + \ell_\alpha$$

The twist values avoid a negligible set (creation of saddle connections), but it is irrelevant as we integrate

A geometric proof of Witten conjecture/Kontsevich theorem

1 - $\forall L \in \mathbb{R}_+^n \quad \mathcal{M}_{g,n} \cong \mathcal{M}_\Sigma^{\mathrm{comb}}(L)$

2 - The class of $\omega_{\mathrm{K}} = \frac{1}{2}\sum_{i=1}^{n} \sum_{\substack{e<e' \\ \text{around } \partial_i \mathbb{G}}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{e'}$ identifies with $\frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i$

3 - Boundary contributions can be ignored (Zvonkine, 06), so that

$$V_{g,n}^{\mathrm{K}}(L) = \int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} \frac{\omega_{\mathrm{K}}^{\wedge(3g-3+n)}}{(3g-3+n)!} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right)$$

4 - The combinatorial Mirzakhani identity realises the constant function 1 on $\mathcal{T}_\Sigma^{\mathrm{comb}}$ as an outcome of geometric recursion

5 - Thanks to combinatorial FN coord. and analog of Wolpert formula

$\Longrightarrow$ topological recursion for $V_{g,n}^{\mathrm{K}}(L)$

For $L \in \mathbb{Z}_+^n$ such that $\sum_i L_i \in 2\mathbb{Z}_+$

we have $\mathcal{M}_\Sigma^{\mathrm{comb},\mathbb{Z}}(L) \subset \mathcal{M}_\Sigma^{\mathrm{comb}}(L)$, set of metric ribbon graphs with integer edge length

so we can define discrete integration

of a function $\Omega_\Sigma \in \mathrm{Fun}(\mathcal{M}_\Sigma^{\mathrm{comb}})$

$$V_{g,n}^{\mathbb{Z}}(L) = \sum_{\mathbf{G} \in \mathcal{M}_\Sigma^{\mathrm{comb},\mathbb{Z}}(L)} \frac{\Omega_\Sigma(\mathbf{G})}{|\mathrm{Aut}\,\mathbf{G}|}$$

Theorem 6 (ABCGLW, 20)

If Ω_Σ is the outcome of GR for initial data $(\Omega_P, B, C, \Omega_T)$

such that $\begin{cases} B(L_1, L_2, \ell) = 0 & \text{if } L_1 + L_2 < \ell \\ C(L_1, \ell, \ell') = 0 & \text{if } L_1 < \ell + \ell' \end{cases}$

then $V_{g,n}^{\mathbb{Z}}(L)$ satisfies topological recursion for initial data $(\Omega_P^{\mathbb{Z}}, B^{\mathbb{Z}}, C^{\mathbb{Z}}, \Omega_T^{\mathbb{Z}})$

where for $X \in C^0(\mathcal{T}_S)$ we set $X^{\mathbb{Z}} = X \cdot \mathbf{1}_{\mathcal{T}_S^{\mathrm{comb},\mathbb{Z}}}$

and replace integrals with sums over $\mathbb{Z}_+$

Remark : the GR sum is finite due to the support condition

Idea of the proof

Last week we have seen that for a seamed pair of pants decomposition the combinatorial Fenchel-Nielsen coordinates give

$$\begin{array}{ccc} \mathcal{T}_{\Sigma}^{\mathrm{comb}}(L) & \longrightarrow & (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \\ \mathbb{G} & \longmapsto & \big(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i)\big)_i \end{array}$$

whose image is the complement of a negligible set Z

Discrete integration over the twist could hit Z

But one proves that any twist on small pairs of pants is well-defined

$\rightsquigarrow$ under the support condition, the fiber has full cardinality ℓ (or $\ell\,\ell'$)

New proof of Norbury's TR for the number of integer points (Norbury, 10)

The combinatorial Mirzakhani identity realises the constant function 1 on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}$ as an outcome of geometric recursion

$$B_{\mathrm{K}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big([L_1 + L_2 - \ell]_+ + [L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+\big)$$

$$C_{\mathrm{K}}(L_1, \ell, \ell') = \frac{1}{L_1}[L_1 - \ell - \ell']_+ \qquad \text{with} \quad [x]_+ = \max(x, 0)$$

The support condition holds

$\Longrightarrow$ Topological recursion for $\left|\mathcal{M}_{\Sigma}^{\mathrm{comb},\mathbb{Z}}\right| = \sum_{\mathbf{G} \in \mathcal{M}_{\Sigma}^{\mathrm{comb},\mathbb{Z}}} \frac{1}{|\mathrm{Aut}(\mathbf{G})|}$

III

Asymptotic count of multicurves

III.1 Asymptotic count of multicurves — Thurston volume of unit ball

Let $\mathrm{MF}_\Sigma \subset \mathrm{MF}^\star_\Sigma$ be the set of measured foliations where $\partial\Sigma$ is a union of sing. leaves

It admits a piecewise linear integral structure and $\dim \mathrm{MF}_\Sigma = 6g - 6 + 2n$

{Integral points of MF_Σ} = M_Σ = {multicurves}

Thurston measure of $A \subset \mathrm{MF}_\Sigma$ $\qquad \mu_{\mathrm{Th}}(A) = \lim_{k\to\infty} \frac{|A \cap k^{-1} M_\Sigma|}{k^{6g-6+2n}}$ if exists

	Hyperbolic	Combinatorial
Length functions	$\mathcal{T}_\Sigma \times \mathrm{MF}_\Sigma \to \mathbb{R}_+$	$\mathcal{T}_\Sigma^{\mathrm{comb}} \times \mathrm{MF}_\Sigma \to \mathbb{R}_+$
Vol. of unit balls	$\mathcal{B}_\Sigma(\sigma) = \mu_{\mathrm{Th}}\big(\{\ell_\sigma \leq 1\}\big)$	$\mathcal{B}_\Sigma^{\mathrm{comb}}(\mathbb{G}) = \mu_{\mathrm{Th}}\big(\{\ell_{\mathbb{G}} \leq 1\}\big)$
Moments on Teichmüller	$V^s\mathcal{B}_{g,n}(L) := \int_{\mathcal{M}_{g,n}(L)} \mathrm{d}\mu_{\mathrm{WP}}(\sigma)\big(\mathcal{B}_\Sigma(\sigma)\big)^s$	$V^s\mathcal{B}_{g,n}^{\mathrm{comb}}(L) := \int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} \mathrm{d}\mu_{\mathrm{K}}(\mathbb{G})\big(\mathcal{B}_\Sigma(\mathbb{G})\big)^s$

Known results for punctured hyperbolic surfaces Σ

- $\mathcal{B}_\Sigma : \mathcal{T}_\Sigma \to \mathbb{R}_+$ is continuous, proper, and

$$c'_{g,n} \prod_{\substack{\gamma \in S_\Sigma \\ \ell_\sigma(\gamma) \leq \epsilon}} \frac{1}{\ell_\sigma(\gamma) \big| \ln(\ell_\sigma(\gamma)) \big|} \leq \mathcal{B}_\Sigma(\sigma) \leq c_{g,n} \prod_{\substack{\gamma \in S_\Sigma \\ \ell_\sigma(\gamma) \leq \epsilon}} \frac{1}{\ell_\sigma(\gamma)}$$

Mirzakhani (07)

$\Longrightarrow$ $V^s \mathcal{B}_{g,n}(0)$ is finite for $s < 2$ and infinite for $s > 2$

- Finer upper bound $\Longrightarrow$ $V^2 \mathcal{B}_{g,n}(0)$ is finite — Arana-Herrera, Athreya (19)

- Relation to Masur-Veech volumes

$$\mathcal{Q}\mathcal{T}_\Sigma \overset{\sim}{\dashrightarrow} \mathrm{MF}_\Sigma \times \mathrm{MF}_\Sigma \overset{\sim}{\dashleftarrow} \mathcal{T}_\Sigma \times \mathrm{MF}_\Sigma$$

$$\mu_{\mathrm{MV}} \qquad \mu_{\mathrm{Th}} \otimes \mu_{\mathrm{Th}} \qquad \mu_{\mathrm{WP}} \otimes \mu_{\mathrm{Th}}$$

Bonahon (96)
Mirzakhani (08)

$$\Longrightarrow \quad V^1 \mathcal{B}_{g,n}(0) = \frac{\mu_{\mathrm{MV}}\big(\mathcal{Q}^1_{g,n}\big)}{2^{4g-2+n} \cdot (6g-6+2n) \cdot (4g-4+n)!}$$

Delecroix, Goujard, Zograf, Zorich (19)
Monin-Telpukhovskiy (19)
Arana-Herrera (19)

Open problem : compute explicitly $\mathcal{B}_\Sigma(\sigma)$ and $\big(V^s \mathcal{B}_{g,n}(L)\big)_{s \neq 1}$

III.2 Asymptotic count of multicurves -- wrt

If $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that $\varphi(\ell) \underset{\ell\to\infty}{=} O(\ell^{-\infty})$

consider multiplicative statistics of lengths of multicurves

$\sigma \in \mathcal{T}_\Sigma$ $$\Omega_{\mathrm{M},\Sigma}[\varphi](\sigma) = \sum_{\gamma\in M'_\Sigma} \prod_{\delta\in\pi_0(\gamma)} \varphi(\ell_\sigma(\delta))$$ (only primitive multicurves)

$$\varphi_s(\ell) = \frac{e^{-s\ell}}{1-e^{-s\ell}} \quad \rightsquigarrow \quad \Omega_{\mathrm{M},\Sigma}[\varphi_s](\sigma) = \sum_{\gamma\in M_\Sigma} e^{-s\ell_\sigma(\gamma)}$$ (all multicurves)

This is related to the counting function by Laplace transform

$$s\int_{\mathbb{R}_+} \mathrm{d}t\, e^{-st}\,\big|\{\gamma\in M_\Sigma \quad | \quad \ell_\sigma(\gamma)\leq t\}\big| = \Omega_{\mathrm{M},\Sigma}[\varphi_s](\sigma)$$

After integration and $s\to 0$ this relates to the Thurston volume by integration

$$s\int_{\mathbb{R}_+}\mathrm{d}t\,e^{-st}\bigg(\int_{\mathcal{M}_\Sigma(L)}\big|\{\gamma\in M_\Sigma \quad | \quad \ell_\sigma(\gamma)\leq t\}\big|\,\mathrm{d}\mu_{\mathrm{WP}}(\sigma)\bigg) \underset{s\to 0}{\sim} \frac{s^{-(6g-6+2n)}}{(6g-6+2n)!}\int_{\mathcal{M}_\Sigma(L)} \mathcal{B}_\Sigma(\sigma)\,\mathrm{d}\mu_{\mathrm{WP}}(\sigma)$$

III.2 Asymptotic count of multicurves -- wrt hyperbolic

Taking as initial data

$$\begin{cases} B_{\mathrm{M}}[\varphi_s](L_1, L_2, \ell) = B_{\mathrm{M}}(L_1, L_2, \ell) + \frac{e^{-s\ell}}{1-e^{-s\ell}} \\ C_{\mathrm{M}}[\varphi_s](L_1, \ell, \ell') = C_{\mathrm{M}}(L_1, \ell, \ell') + \frac{B_{\mathrm{M}}(L_1,\ell,\ell')\, e^{-s\ell}}{1-e^{-s\ell}} + \frac{B_{\mathrm{M}}(L_1,\ell',\ell)\, e^{-s\ell'}}{1-e^{-s\ell'}} + \frac{e^{-s\ell}}{1-e^{-s\ell}}\, \frac{e^{-s\ell'}}{1-e^{-s\ell'}} \end{cases}$$

- $$s \int_{\mathbb{R}_+} \mathrm{d}t\, e^{-st} \left| \{\gamma \in M_\Sigma \quad | \quad \ell_\sigma(\gamma) \leq t\} \right| = \Omega_{\mathrm{M},\Sigma}[\varphi_s](\sigma)$$

  is computed by geometric recursion (Theorem 3)

- $$s \int_{\mathbb{R}_+} \mathrm{d}t\, e^{-st} \left(\int_{\mathcal{M}_\Sigma(L)} \left| \{\gamma \in M_\Sigma \quad | \quad \ell_\sigma(\gamma) \leq t\} \right| \mathrm{d}\mu_{\mathrm{WP}}(\sigma) \right)$$

  is computed by topological recursion (Theorem 4)

- The $s \to 0$ limit of TR can be studied by the change of integration variable $\ell \mapsto \ell/s$ and recalling $\lim_{s \to 0} X_{\mathrm{M}}(\vec{\ell}/s) = X_{\mathrm{K}}(\vec{\ell})$

Theorem 7 **(ABCDGLW, 19 | ABCGLW 20)**

The hyperbolic GR for initial data

$$\begin{cases} B_{\mathrm{K}}[\varphi_1](L_1,L_2,\ell) = B_{\mathrm{K}}(L_1,L_2,\ell) + \frac{e^{-\ell}}{1-e^{-\ell}} \\ C_{\mathrm{K}}[\varphi_1](L_1,\ell,\ell') = C_{\mathrm{K}}(L_1,\ell,\ell') + \frac{B_{\mathrm{K}}(L_1,\ell,\ell')\,e^{-\ell}}{1-e^{-\ell}} + \frac{B_{\mathrm{K}}(L_1,\ell',\ell)\,e^{-\ell'}}{1-e^{-\ell'}} + \frac{e^{-\ell}}{1-e^{-\ell}}\,\frac{e^{-\ell'}}{1-e^{-\ell'}} \end{cases}$$

produces a function $\Omega^{\mathrm{hyp}}_{\mathrm{K},\Sigma}[\varphi_1] \in C^0(\mathcal{T}_\Sigma)^{\mathrm{Mod}^\partial_\Sigma}$ whose integral on $(\mathcal{M}_\Sigma(L), \mathrm{d}\mu_{\mathrm{WP}})$

is computed by TR and satisfies

$$\lim_{L\to 0^+} V\Omega^{\mathrm{hyp}}_{\mathrm{K},g,n}[\varphi_1](L) = \frac{V\mathcal{B}_{g,n}(L)}{(6g-6+2n)!}$$ independent of L

The same reasoning can be done for the counting wrt combinatorial length

$$s \int_{\mathbb{R}_+} \mathrm{d}t\, e^{-st} \left(\int_{\mathcal{M}_\Sigma^{\mathrm{comb}}(L)} \big| \{ \gamma \in M_\Sigma \quad | \ell_{\mathbb{G}}(\gamma) \leq t \} \big| \, \mathrm{d}\mu_{\mathrm{K}}(\mathbb{G}) \right) \underset{s \to 0}{\sim} \frac{V\mathcal{B}_\Sigma^{\mathrm{comb}}(L)}{(6g-6+2n)!\, s^{6g-6+2n}}$$

Theorem 7bis **(ABCDGLW, 19 | ABCGLW 20)**

The combinatorial GR for initial data

$$\begin{cases} B_{\mathrm{K}}[\varphi_1](L_1, L_2, \ell) = B_{\mathrm{K}}(L_1, L_2, \ell) + \frac{e^{-\ell}}{1-e^{-\ell}} \\ C_{\mathrm{K}}[\varphi_1](L_1, \ell, \ell') = C_{\mathrm{K}}(L_1, \ell, \ell') + \frac{B_{\mathrm{K}}(L_1,\ell,\ell')\, e^{-\ell}}{1-e^{-\ell}} + \frac{B_{\mathrm{K}}(L_1,\ell',\ell)\, e^{-\ell'}}{1-e^{-\ell'}} + \frac{e^{-\ell}}{1-e^{-\ell}} \frac{e^{-\ell'}}{1-e^{-\ell'}} \end{cases}$$

produces a function $\Omega_{\Sigma,\mathrm{K}}[\varphi_1] \in C^0(\mathcal{T}_\Sigma^{\mathrm{comb}})^{\mathrm{Mod}_\Sigma^\partial}$ whose integral on $(\mathcal{M}_\Sigma^{\mathrm{comb}}(L), \mathrm{d}\mu_{\mathrm{K}})$

is computed by TR and satisfies

$$\lim_{L \to 0^+} V\Omega_{\mathrm{K},g,n}[\varphi_1](L) = \frac{V\mathcal{B}_{g,n}^{\mathrm{comb}}(L)}{(6g-6+2n)!}$$

independent of L

III.3 Asymptotic count of multicurves -- wrt combinatorial

Same TR formula (though different GR -- hyperbolic or combinatorial)

$\Longrightarrow \quad V\mathscr{B}_{g,n}(L) = V\mathscr{B}^{\mathrm{comb}}_{g,n}(L)$ independent of L

This allows computing them as the constant term of a family of functions (polynomials) satisfying TR

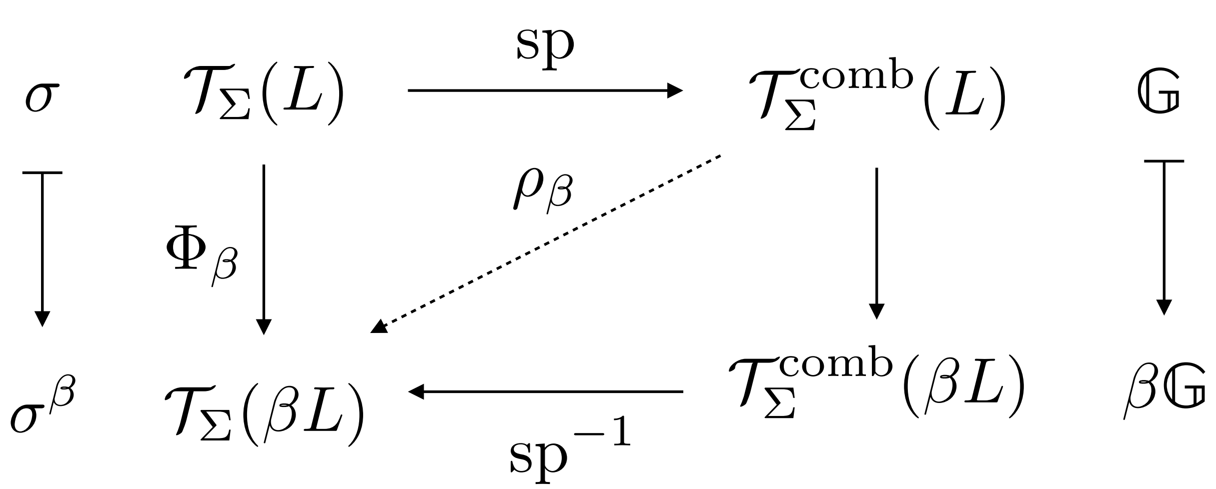
The Masur-Veech volumes are insensitive to the model used for Teichmüller space

Theorem 8 **(B, Charbonnier, Delecroix, Giacchetto, Wheeler, to appear)**

$V^s\mathscr{B}^{\mathrm{comb}}_{g,n}(L)$ is finite

iff $\quad s < s^*_{g,n} \leq 2$

g/n	1	2	3	4	5	$\geq$ 6
0			∞	2	2	$\frac{4}{3} + \frac{1}{2(\lfloor n/2 \rfloor - 2)}$
1	2	$\frac{4}{3}$				
2	$\frac{4}{3}$	$1 + \frac{1}{3(2g-1)}$				
≥ 3	$1 + \frac{1}{3(2g-3)}$					



Jacobian

$$J_\beta := \frac{1}{\beta^{6g-6+2n}} \frac{\rho_\beta^* \mathrm{d}\mu_{\mathrm{WP}}}{\mathrm{d}\mu_{\mathrm{K}}}$$

- By Lemma 5 $\lim_{\beta \to \infty} \beta^{6g-6+2n} \rho_\beta^* \mathcal{B}_\Sigma = \mathcal{B}_\Sigma^{\mathrm{comb}}$ uniform cv. on thick parts of $\mathcal{T}_\Sigma^{\mathrm{comb}}$
- By Mondello (09) $\lim_{\beta \to \infty} J_\beta = 1$

Fatou lemma $\Longrightarrow$ $V^s \mathcal{B}_{g,n}^{\mathrm{comb}}(L) \leq \liminf_{\beta \to \infty} \dfrac{V^s \mathcal{B}_{g,n}(\beta L)}{\beta^{(6g-6+2n)(s-1)}}$

- For $s \geq s_{g,n}^*$, LHS infinite $\Rightarrow$ anomalous scaling of $V^s \mathcal{B}_{g,n}(L)$ for large length
- By Lemma 10, for $s = 1$, both sides are equal (independent of L thus β)

Miss a uniform 'integrable' bound on J_β to study equality for $s < s_{g,n}^*$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Thank you for your attention !

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