BISTRO seminar

Geometry of combinatorial moduli spaces and multicurve counts

based on joint works with
Andersen, Charbonnier, Delecroix, Giacchetto, Lewanski, Wheeler : math.GT/1905.10352
ACGLW to appear
CDGW to appear

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Σ will usually denote a smooth bordered surface
oriented, connected (unless specified), genus g
n labeled boundaries $\partial_1\Sigma, \ldots, \partial_n\Sigma$
stable : $2 - 2g - n < 0$

I. Geometry of the combinatorial Teichmüller space

II. Flowing from hyperbolic to combinatorial

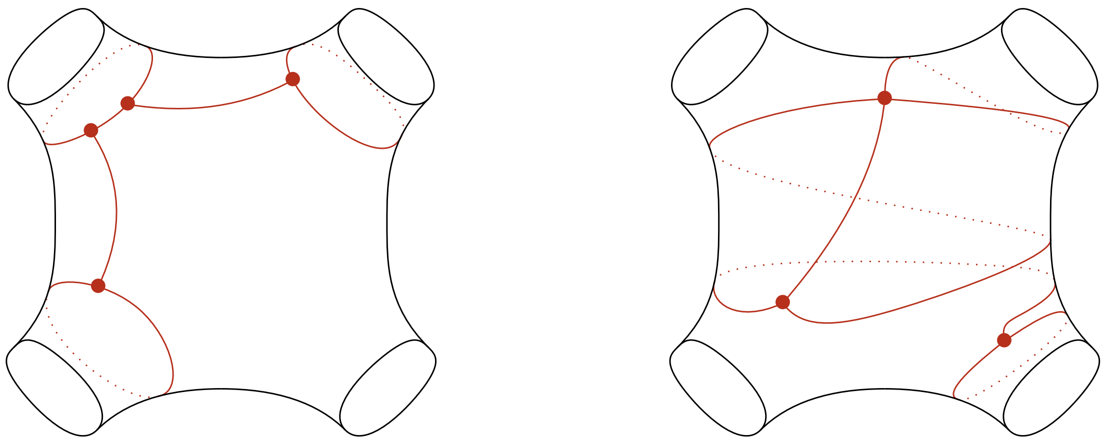
III. McShane, Mirzakhani, multicurves

IV. Thurston volume of unit balls

I

Geometry of the combinatorial Teichmüller space

A **ribbon graph** is a graph with
- the data of a cyclic order at each vertex
- vertices have valency ≥ 3
- faces are labeled from 1 to n



Combinatorial Teichmüller space

$$\mathcal{T}_\Sigma^{\mathrm{comb}} = \left\{ \begin{array}{l} \text{isotopy class of proper embeddings of metric ribbon graphs} \\ \mathbb{G} \overset{f}{\hookrightarrow} \Sigma \quad \text{such that } \Sigma \text{ retracts onto } f(\mathbb{G}) \text{ and labels agree} \end{array} \right\} \circlearrowleft \mathrm{Mod}_\Sigma^\partial$$

pure mapping class group

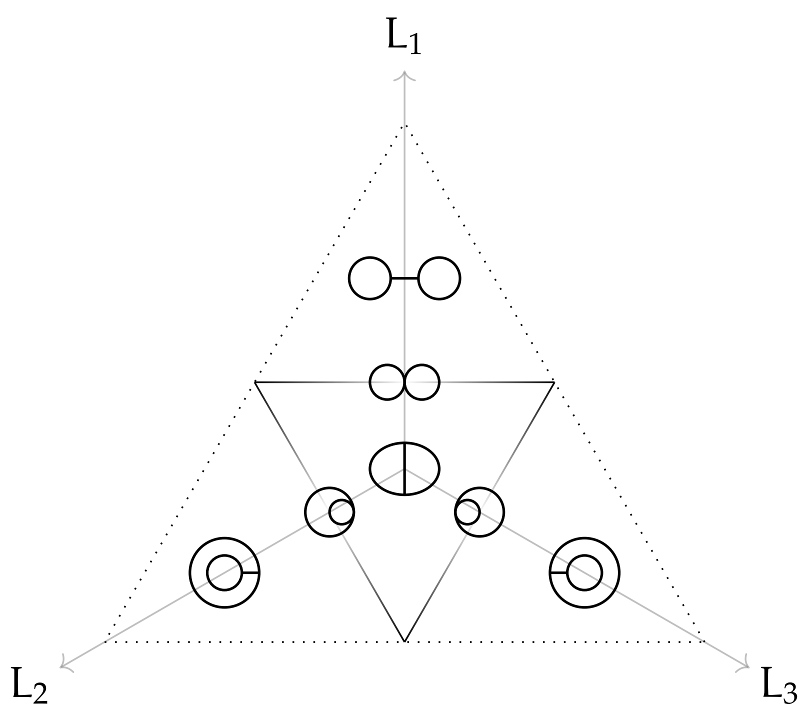
Combinatorial moduli space

$$\mathcal{M}_\Sigma^{\mathrm{comb}} = \frac{\mathcal{T}_\Sigma^{\mathrm{comb}}}{\mathrm{Mod}_\Sigma^\partial} = \bigcup_{\substack{G \text{ ribbon graph} \\ \text{type } (g,n)}} \frac{\mathbb{R}_+^{E(G)}}{\mathrm{Aut}\, G}$$

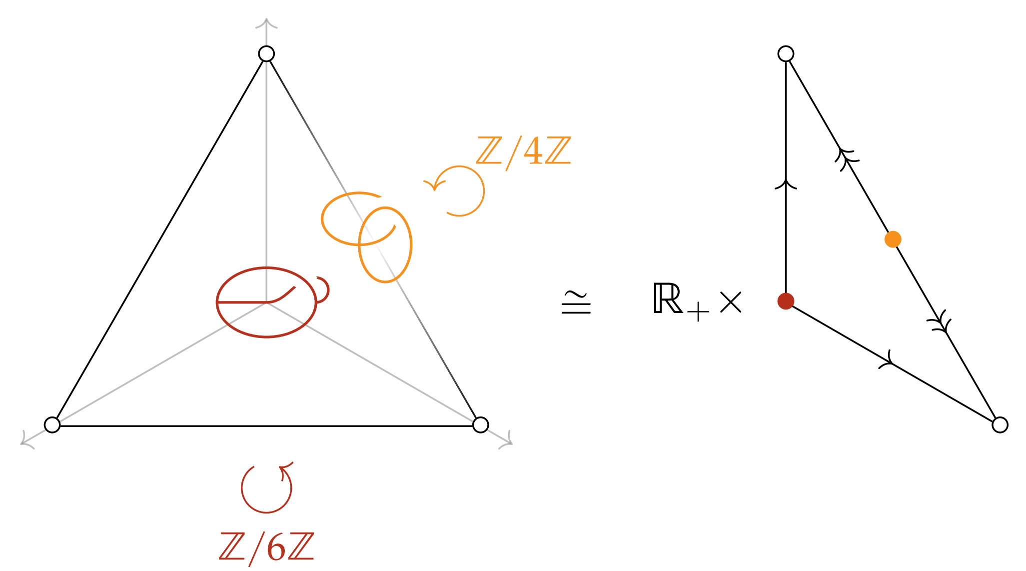
I.1 Combinatorial Teichmüller space — Definitions

Examples of combinatorial moduli spaces

$(g,n) = (0,3)$



$(g,n) = (1,1)$

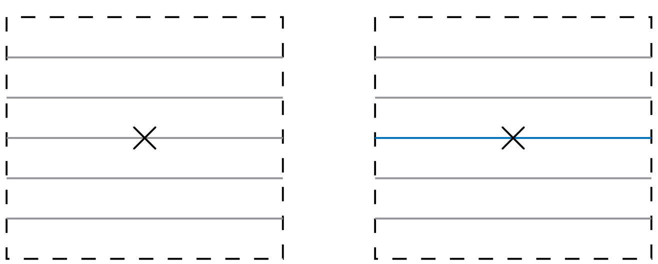


$\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$, $\mathcal{M}_{\Sigma}^{\mathrm{comb}}(L)$ loci with fixed boundary lengths $L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n$

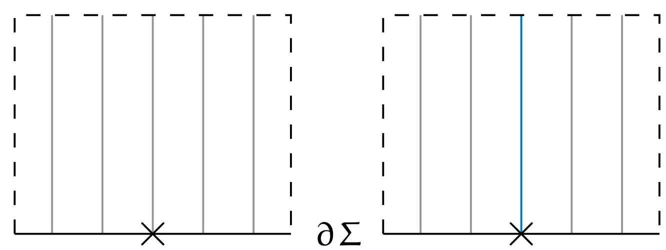
They are not smooth spaces, but rather polytopal complexes

The combinatorial Teichmüller space has an equivalent description by measured foliations

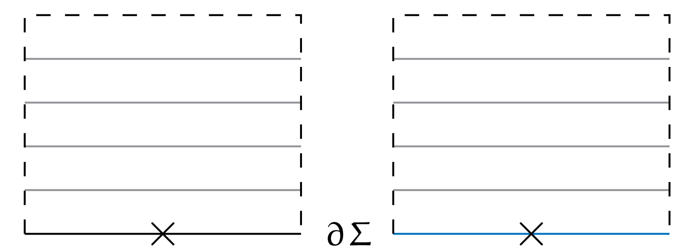
$$\mathrm{MF}^{\star}_{\Sigma} = \left\{ (\mathcal{F}, \mu) \;\middle|\; \begin{array}{ll} \mathcal{F} & \text{foliation with isolated singularities} \\ \mu & \text{transverse invariant measure} \end{array} \right\} \Big/ \begin{array}{l} \text{isotopies} \\ \text{Whitehead moves} \end{array}$$



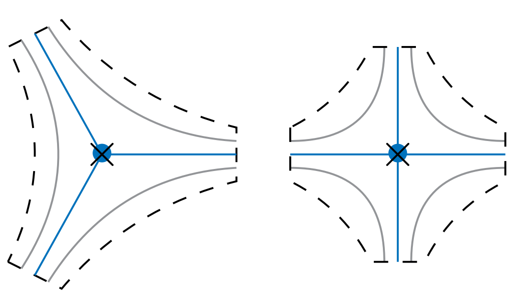
(a) Internal regular point.



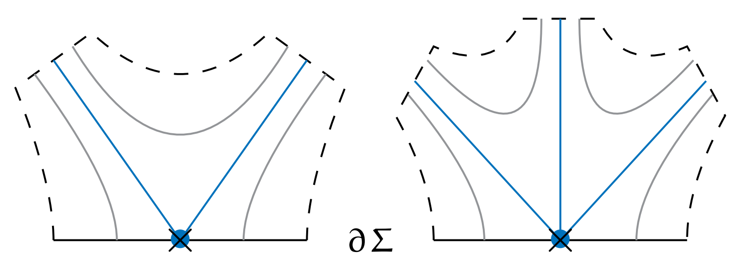
(b) Regular point at the boundary of transverse type.



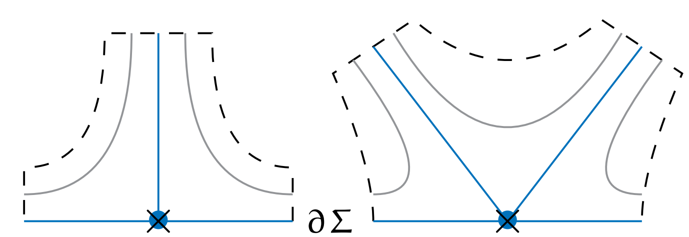
(c) Regular point at the boundary of parallel type.



(d) Internal singular point.



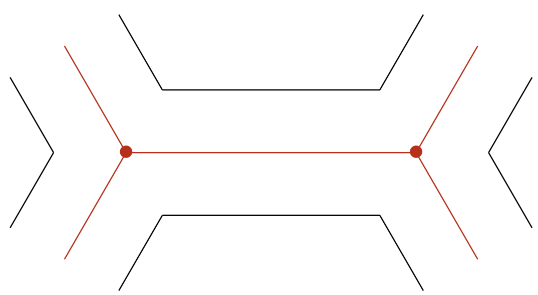
(e) Singular point at the boundary of transverse type.



(f) Singular point at the boundary of parallel type.

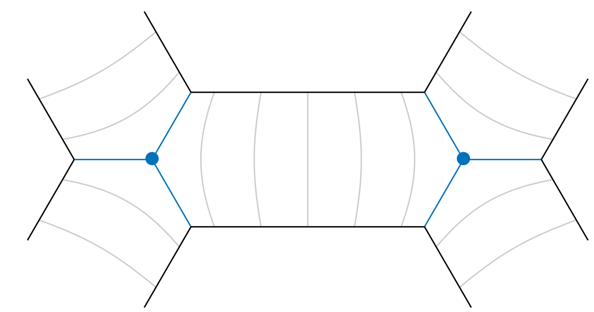
I.1 Combinatorial Teichmüller space — Definitions

The combinatorial Teichmüller space has an equivalent description by measured foliations

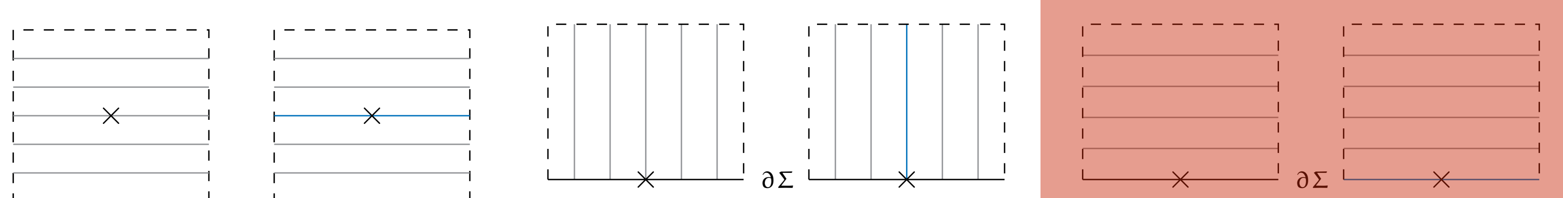


$$\mathcal{T}_\Sigma^{\mathrm{comb}} \hookrightarrow \mathrm{MF}_\Sigma^\star$$

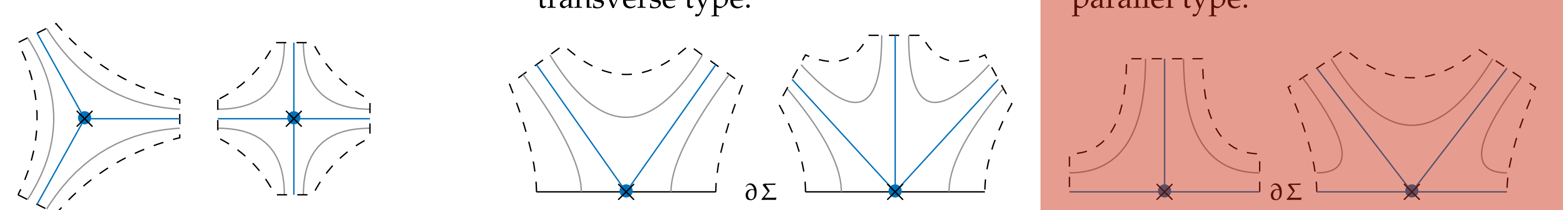
homeomorphism onto its image



The image is the set of [measured foliations] where
- leaves are transverse to $\partial\Sigma$
- no saddle connections



(a) Internal regular point. (b) Regular point at the boundary of transverse type. (c) Regular point at the boundary of parallel type.



(d) Internal singular point. (e) Singular point at the boundary of transverse type. (f) Singular point at the boundary of parallel type.

If $\gamma \in S_{\Sigma}^{\bullet}$ = {homotopy classes of simple closed curves}

we have a combinatorial length functions $\ell(\gamma) \, : \, \mathcal{T}_{\Sigma}^{\mathrm{comb}} \to \mathbb{R}_{+}$ (continuous)

- sum of edge lengths along the non-backtracking rep. on the graph
- intersection number with the measured foliation

Kontsevich 2-form on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$

defined on cells, $\mathrm{Mod}_{\Sigma}^{\partial}$ - invariant

$$\omega_{\mathrm{K}} = \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{e<e' \\ \text{around } \partial_i \Sigma}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{e'}$$

Lemma (Kontsevich, 91) ω_{K} is non-degenerate on cells corresponding to ribbon graphs with vertices of odd valency only

Kontsevich 2-form on $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$
defined on cells, $\mathrm{Mod}_\Sigma^\partial$ - invariant

$$\omega_{\mathrm{K}} = \frac{1}{2}\sum_{i=1}^{n} \sum_{\substack{e<e' \\ \text{around } \partial_i\Sigma}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{e'}$$

Introduced by Kontsevich in his proof of Witten's conjecture

1 - $\forall L \in \mathbb{R}_+^n \quad \mathcal{M}_{g,n} \cong \mathcal{M}_\Sigma^{\mathrm{comb}}(L)$

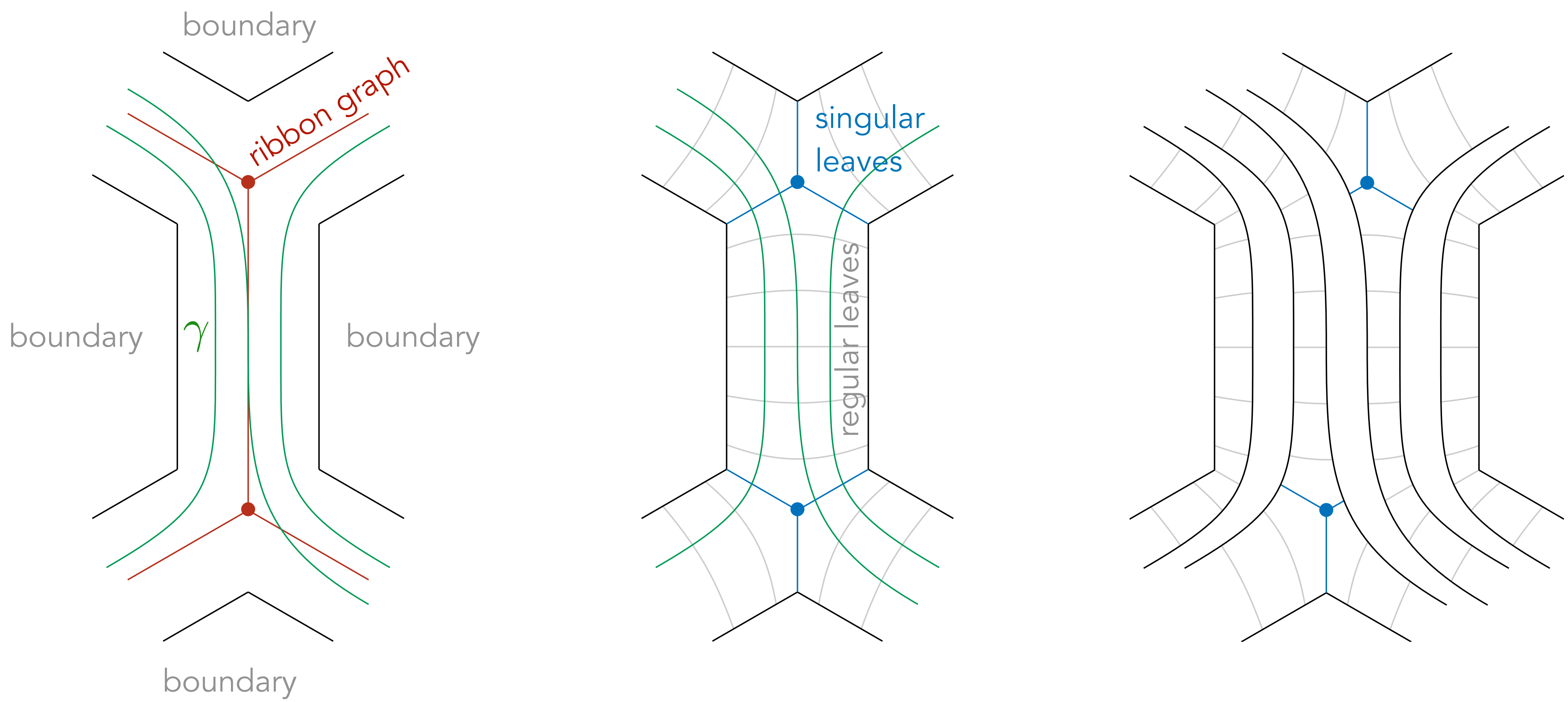
2 - $V_\Sigma^{\mathrm{K}}(L) := \displaystyle\int_{\mathcal{M}_\Sigma^{\mathrm{comb}}(L)} \frac{\omega_{\mathrm{K}}^{\wedge d_\Sigma}}{d_\Sigma!} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right)$

3 - matrix model representation $\rightsquigarrow$ KdV hierarchy and Virasoro constraints

Dijkgraaf, Vafa, Verlinde (91)

Although $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$ and $\mathcal{T}_\Sigma(L)$ are homeomorphic, they carry different geometry reflected in their respective symplectic forms ω_{K} and ω_{WP}

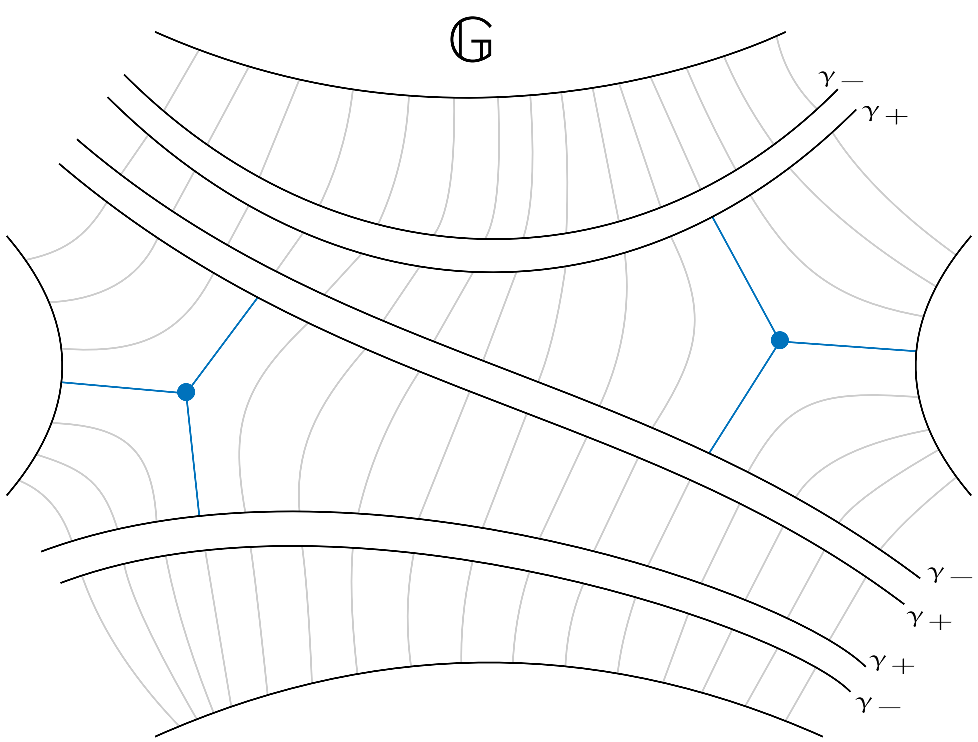
If γ is an oriented simple closed curve, we can cut $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$ along γ and obtain $\mathbb{G}_{|\Sigma-\gamma} \in \mathcal{T}_{\Sigma-\gamma}^{\text{comb}}$

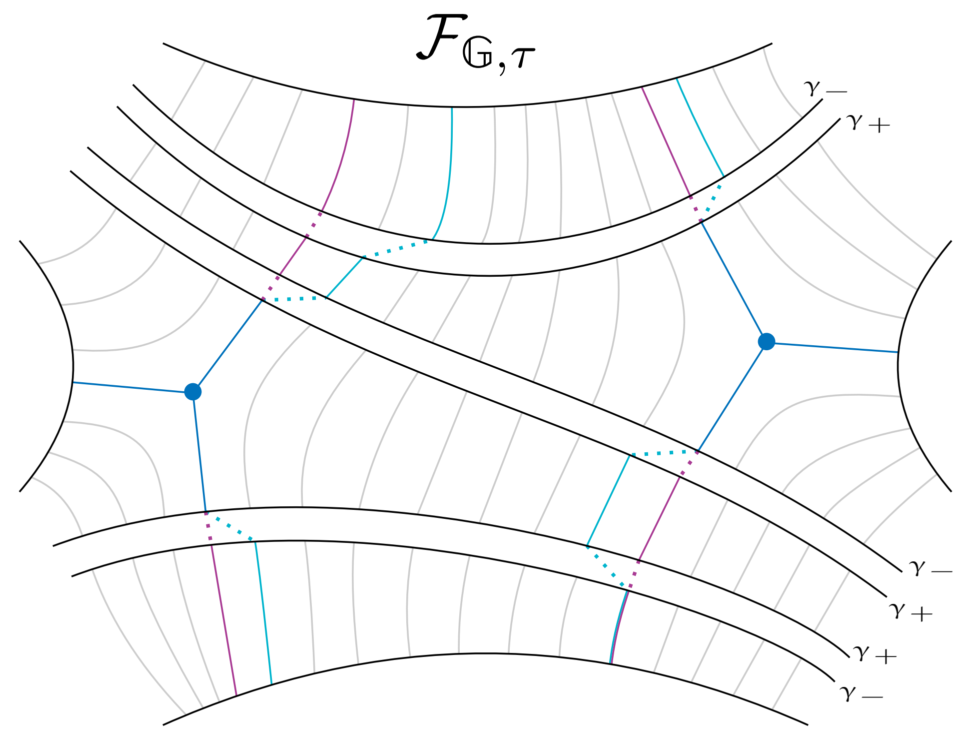


$\mathbb{G}$ $\quad$ $\mathbb{G}_{|\Sigma-\gamma}$

If Σ' is a surface (possibly disconnected) with a choice of two boundaries $\partial_{\pm}\Sigma'$ and points $p_{\pm} \in \partial_{\pm}\Sigma'$ and $\tau \in \mathbb{R}$ and $\mathbb{G} \in \mathcal{T}_{\Sigma'}^{\mathrm{comb}}$

we can defined a glued surface and $\mathcal{F}_{\mathbb{G},\tau} \in \mathrm{MF}_{\Sigma}^{\star}$ by sliding p_- of the amount τ

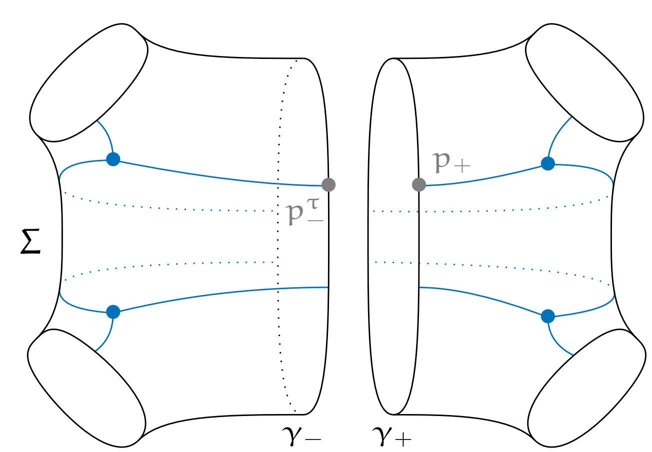


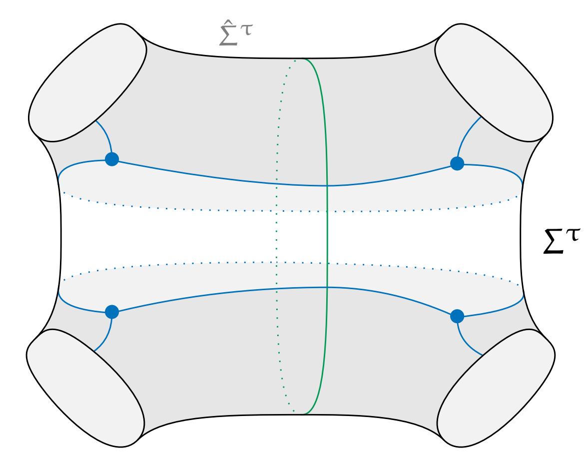


However, $\mathcal{F}_{\mathbb{G},\tau}$ may have saddle connections

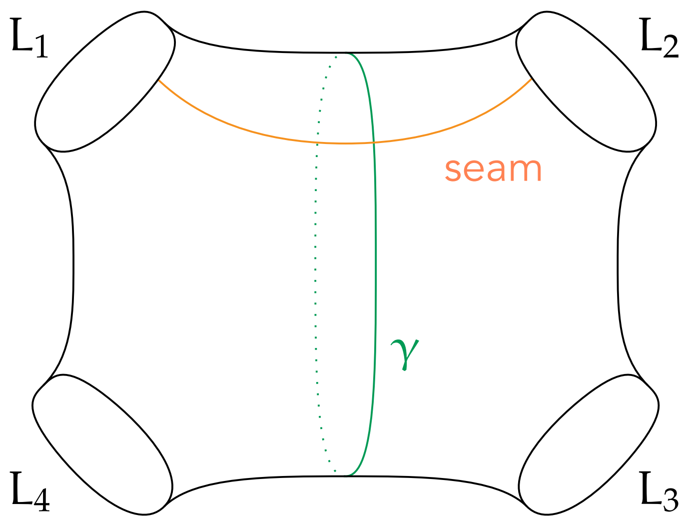
Lemma 1 $\mathcal{F}_{\mathbb{G},\tau} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}}$

except for countably many τ





Take a seamed pair of pants decomposition of Σ



(g,n) = (0,4) example

We have a continuous map

$$\begin{array}{ccc} \mathcal{T}_{\Sigma}^{\mathrm{comb}}(L) & \longrightarrow & (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \\ \mathbb{G} & \longmapsto & \big(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i)\big)_i \end{array}$$

Theorem 2 **(ABCGLW, 20)**

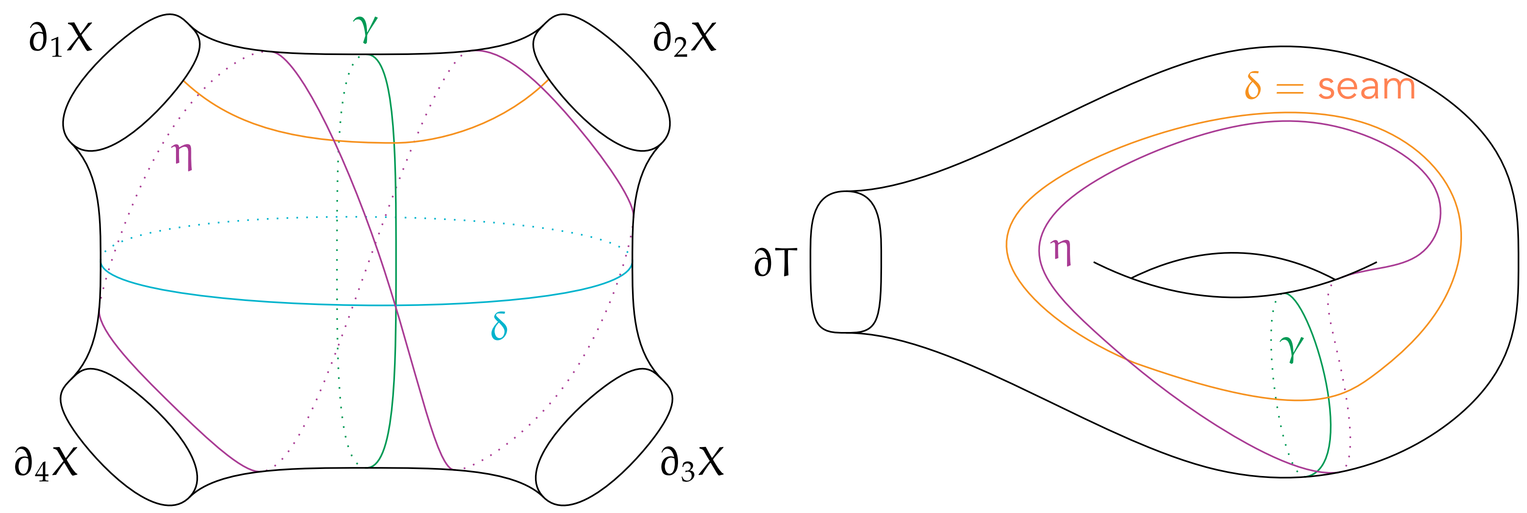
This is an homeomorphism onto its image, which is open dense with complement of zero measure

$\rightsquigarrow$ Combinatorial Fenchel-Nielsen coordinates

For each γ_i in the pair of pants decomposition, define

δ_i determined by the seam

η_i image of δ_i by a positive Dehn twist along γ_i



Combinatorial (9g - 9 + 3n)-theorem **(ABCGLW, 20)**

$$\begin{array}{ccc} \mathcal{T}_{\Sigma}^{\mathrm{comb}}(L) & \longrightarrow & \mathbb{R}_{+}^{9g-9+3n} \\ \mathbb{G} & \longmapsto & \big(\ell_{\mathbb{G}}(\gamma_i), \ell_{\mathbb{G}}(\delta_i), \ell_{\mathbb{G}}(\eta_i)\big)_i \end{array}$$

is a homeomorphism onto its image

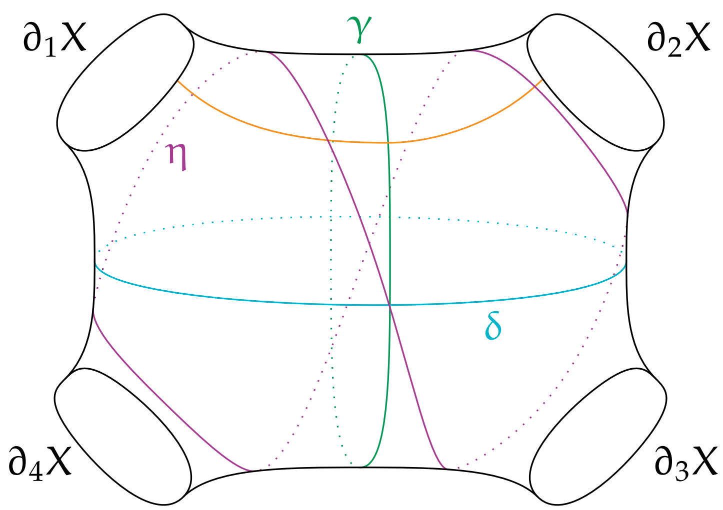
In other words, one can express the twists in terms of lengths of simple closed curves

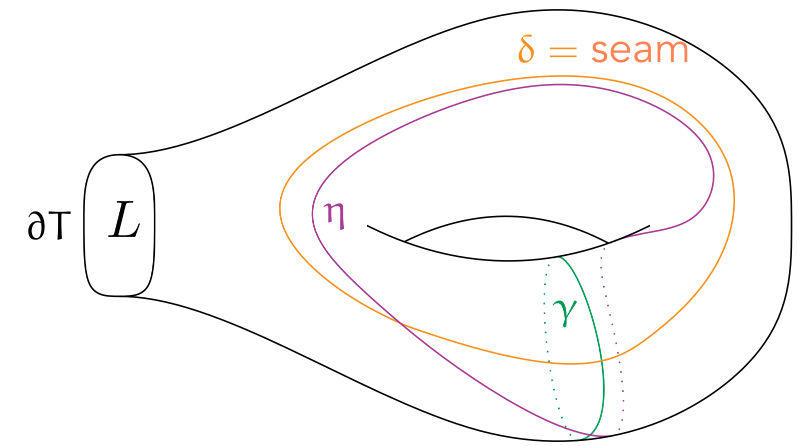
I.3 Combinatorial Teichmüller space — FN coordinates

For each γ_i in the pair of pants decomposition, define

δ_i determined by the seam

η_i image of δ_i by a positive Dehn twist along γ_i





Idea of the proof

- In (1,1) : 4 cases (top cells for the pair of pants), where one checks

$$\begin{cases} \ell(\delta) = |\tau(\gamma)| + \left[\frac{L}{2} - \ell(\gamma)\right]_+ \\ \ell(\eta) = |\tau(\gamma) + \ell(\gamma)| + \left[\frac{L}{2} - \ell(\gamma)\right]_+ \end{cases}$$

inverted as $\tau(\gamma) = \frac{1}{2\ell(\gamma)}\Big(\ell(\eta) - \left[\frac{L}{2} - \ell(\gamma)\right]_+\Big)^2 - \frac{1}{2\ell}\Big(\ell(\delta) - \left[\frac{L}{2} - \ell(\gamma)\right]_+\Big)^2 - \frac{\ell(\gamma)}{2}$

- In (0,4) : 4 top cells for each pair of pants $\to$ 16 cases to discuss

Theorem 4 **(ABCGLW, 20)** For any seamed pair of pants decomposition in each open cell

$$\omega_K = \sum_{i=1}^{3g-3+n} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i$$

$\rightsquigarrow$ combinatorial analog of Wolpert's formula (83)
for Weil-Petersson symplectic form wrt. hyperbolic length/twists

Idea of the proof

Compute the vector field ∂_{τ_i} in terms of edge lengths along γ_i (sliding)

Check it is the hamiltonian vector field for ℓ_i

II

Flowing from hyperbolic to combinatorial

II.1 Flowing from hyperbolic to combinatorial — Identification

For $L \in \mathbb{R}_+$, the Teichmüller space of the bordered surface Σ can be described as

$$\mathcal{T}_\Sigma(L) = \left\{ \begin{array}{c} \text{hyperbolic metrics } \sigma \text{ on } \Sigma \\ \text{geodesic boundaries : } \ell_\sigma(\partial_i \Sigma) = L_i \end{array} \right\} \Big/ \mathrm{Diff}_0(\Sigma) \quad \circlearrowleft \mathrm{Mod}_\Sigma^\partial$$

- It is a smooth space, equipped with Weil-Petersson symplectic form ω_{WP} which is $\mathrm{Mod}_\Sigma^\partial$-invariant
- It admits Fenchel-Nielsen coordinates $\mathcal{T}_\Sigma(L) \simeq (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$

  and we have Wolpert's formula $\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3+n} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i$
- There is a (9g - 9 + 3n)-theorem

II.1 Flowing from hyperbolic to combinatorial — Identification

The spine of a hyperbolic metric σ is the locus of points in Σ equidistant from two boundaries

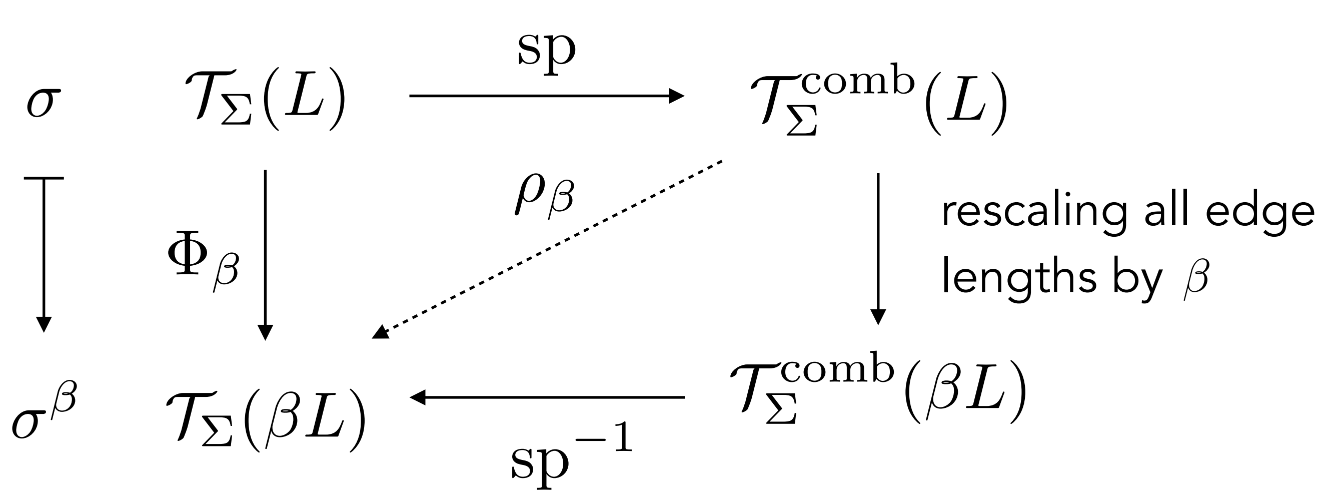
Lemma (Luo 07, Mondello 09)

$$\begin{array}{cccc} \mathrm{sp} : & \mathcal{T}_{\Sigma} & \longrightarrow & \mathcal{T}_{\Sigma}^{\mathrm{comb}} \\ & \sigma & \longmapsto & \mathrm{sp}_{\sigma}(\Sigma) \end{array}$$

is a $\mathrm{Mod}_{\Sigma}^{\partial}$-equivariant homeomorphism

The inverse is poorly understood …

Bowditch-Epstein flow (88)

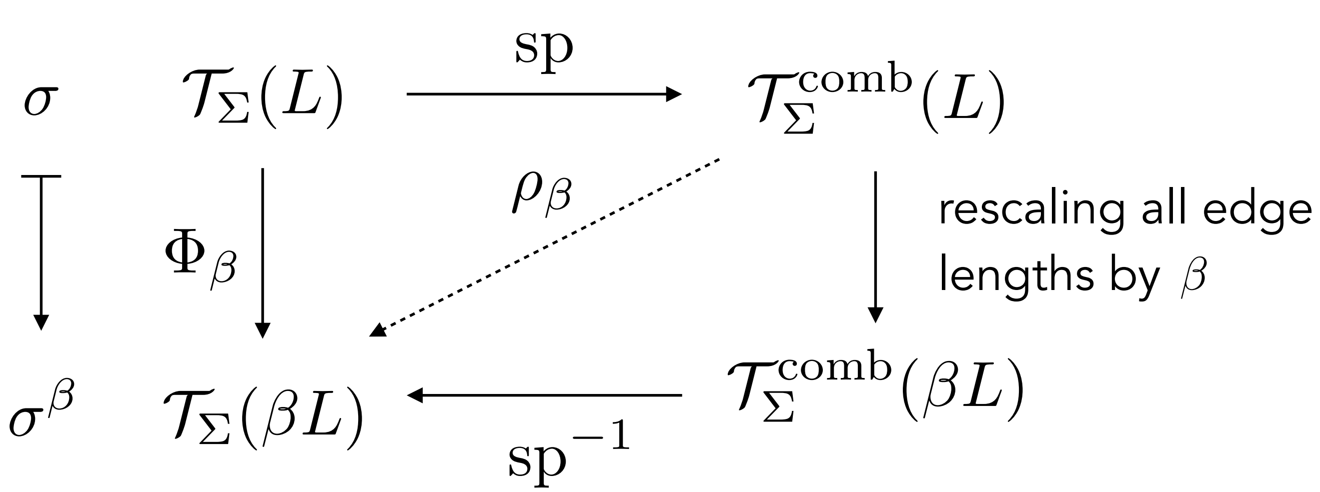


The map ρ_β is not explicit …

II.2 Flowing from hyperbolic to combinatorial — Convergence

Combinatorial geometry is hyperbolic geometry with large boundary lengths

Bowditch-Epstein flow (88)



Theorem (Mondello 09, Do 10) When $\beta \to \infty$

As metric spaces $(\Sigma, \beta^{-1}\sigma^\beta) \to \mathrm{sp}(\sigma)$ in Gromov-Hausdorff sense

$\forall \gamma \in S_{\Sigma}^{\bullet}$ $\beta^{-1}\ell_{\sigma^\beta}(\gamma) \to \ell_{\mathrm{sp}(\sigma)}(\gamma)$ pointwise for $\sigma \in \mathcal{T}_{\Sigma}(L)$

Poisson structure $\beta^2 \rho_\beta^* \Pi_{\mathrm{WP}} \to \Pi_{\mathrm{K}}$ pointwise in $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$

Lemma 5 **(ABCGLW, 20)**

For any $\epsilon > 0$, there is $C_{\epsilon,g,n} > 0$ such that for $\beta \geq \beta_{\epsilon,g,n}$

for any simple closed curve γ and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}}$ with $\mathrm{sys}_{\mathbb{G}} \geq \epsilon$

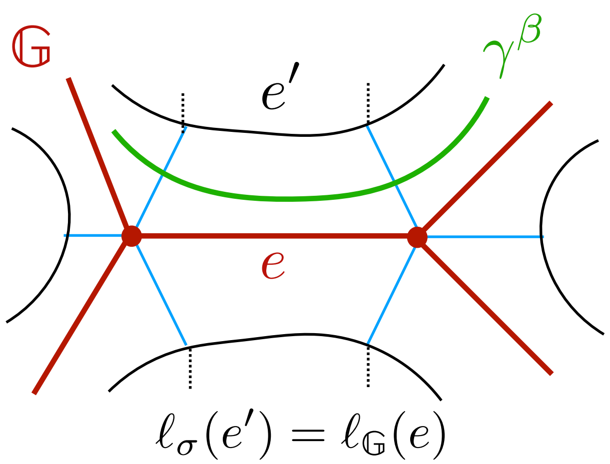
$$\frac{\ell_{\sigma^\beta}(\gamma)}{\beta + C_{\epsilon,g,n}} \leq \ell_{\mathbb{G}}(\gamma) \leq \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \qquad \text{where } \sigma = \mathrm{sp}^{-1}(\mathbb{G})$$

Idea of the proof

- (Do, 10) Upper bound OK, and lower bound

$$\frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \leq \ell_{\mathbb{G}}(\gamma) + 2|E(\gamma)|\,\frac{r_\beta}{\beta} \qquad r_\beta = \max d_{\sigma^\beta}\big(\partial\Sigma, V(\mathrm{sp}(\sigma))\big)$$

$$E(\gamma) = \{\text{edges along } \gamma\}$$



- No cycle shorter than ϵ $\Rightarrow$ $|E(\gamma)| \leq \dfrac{c\,\ell_{\mathbb{G}}(\gamma)}{\epsilon}$

- Area bound $\mathrm{injrad}_{\sigma^\beta} = \max\left(\tfrac{1}{2}\mathrm{sys}_{\sigma^\beta}, \sup_{q\in\Sigma} d_{\sigma^\beta}(q,\partial\Sigma)\right) \leq c'$

- $\mathrm{sys}_{\sigma^\beta} \geq \beta\epsilon$ from upper bound, hence $r_\beta \leq c'$ for β large enough

Proposition 6 **(ABCGLW, 20)**

For each seamed pair of pants decomposition and compact $K \subset \mathcal{T}_{\Sigma}^{\mathrm{comb}}$

there exists $C'_K > 0$ such that, for $\beta \geq \beta_K$

$$\forall i \quad \left| \frac{\tau_i(\sigma^\beta)}{\beta} - \tau_i(\mathbb{G}) \right| \leq \frac{C'_K}{\beta} \qquad \text{where} \quad \sigma = \mathrm{sp}^{-1}(\mathbb{G})$$

Idea of the proof

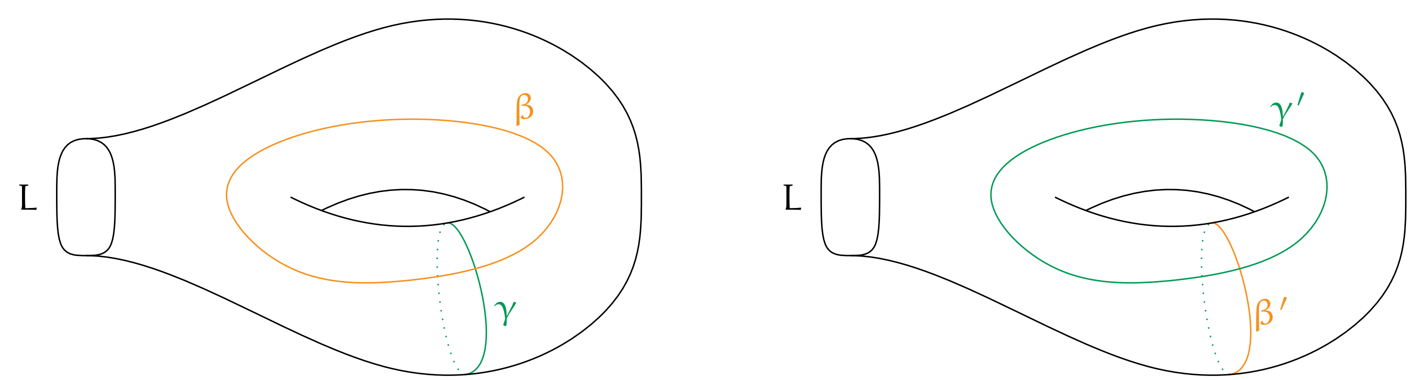
- Use hyp. (9g - 9 + 3n)-theorem to write $\tau_i(\sigma^\beta)$ in terms of hyp. lengths for σ^β
- Prove commensurable upper and lower bounds in terms of comb. lengths for $\mathbb{G}$
- Use comb. (9g - 9 + 3n)-theorem in reverse to write bounds solely with $\tau_i(\mathbb{G})$

Change of pairs of pants $\rightsquigarrow$ transformation of FN coordinates

Can be computed from the $\mathrm{SL}_2(\mathbb{R})$-character variety perspective on $\mathcal{T}_\Sigma(L)$ (Okai, 92)

For σ^β with $\beta \to \infty$, they get tropicalized. (character variety perspective for comb. ??)

Example: flip in torus



$$\begin{cases} \cosh^2\left(\frac{\ell'}{2}\right) = \dfrac{\cosh\left(\frac{\tau}{2}\right)}{\sinh\left(\frac{\ell}{2}\right)}\sqrt{\dfrac{\cosh\left(\frac{L}{2}\right)+\cosh(\ell)}{2}} \\ \cosh\left(\frac{\tau'}{2}\right) = \cosh\left(\frac{\ell}{2}\right)\sqrt{\dfrac{\cosh^2\left(\frac{\tau}{2}\right)\left(\cosh\left(\frac{L}{2}\right)+\cosh(\ell)\right) - 2\sinh^2\left(\frac{\ell}{2}\right)}{\cosh^2\left(\frac{\tau}{2}\right)\left(\cosh\left(\frac{L}{2}\right)+\cosh(\ell)\right) + \sinh^2\left(\frac{\ell}{2}\right)\left(\cosh\left(\frac{L}{2}\right)-1\right)}} \\ \mathrm{sgn}(\tau') = -\mathrm{sgn}(\tau) \end{cases} \Longrightarrow \begin{cases} \ell' = |\tau| + \left[\frac{L}{2} - \ell\right]_+ \\ |\tau'| = -\mathrm{sgn}(\tau)\left|\ell - \left[\frac{L}{2} - \ell'\right]_+\right| \end{cases}$$

hyperbolic

combinatorial

Corollary 7 **(ABCGLW, 20)**

$\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$ admits a piecewise linear structure (given by comb. FN coordinates)

III

McShane, Mirzakhani, multicurves

Summary

1. there is an analogue of Mirzakhani-McShane identity on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}$
2. integrating it (using Wolpert comb. formula) gives a recursion for

$$V_{\Sigma}^{\mathrm{K}}(L) := \int_{\mathcal{M}_{\Sigma}^{\mathrm{comb}}(L)} \frac{\omega_{\mathrm{K}}^{\wedge d_{\Sigma}}}{d_{\Sigma}!} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right)$$

(geometric proof of Witten's conjecture - Virasoro part)

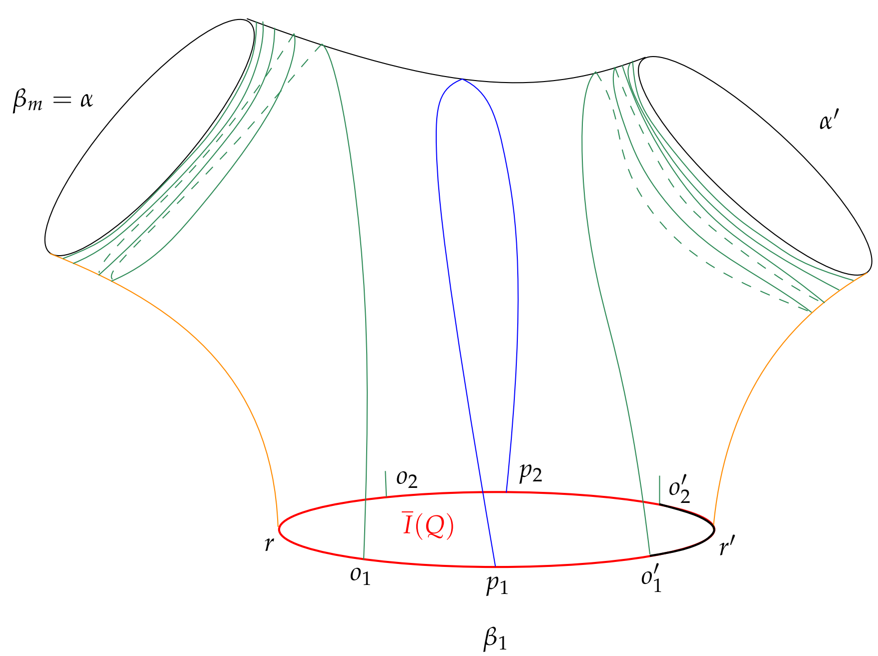
3. this generalises to statistics of multicurves wrt. hyp. or comb. lengths and fits in a general formalism geometric recursion / topological recursion

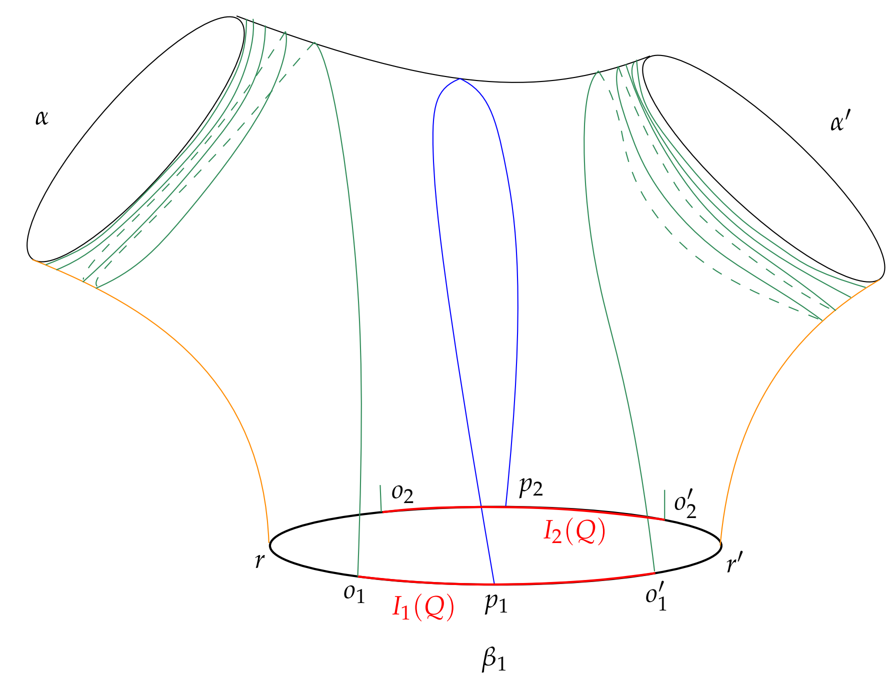
Andersen, B., Orantin (17)

Mirzakhani (07) established a partitition of unity on $\mathcal{T}_\Sigma(L)$
generalizing a identity of McShane (91) for the punctured torus

Strategy in a hyperbolic surface (Σ, σ) with $2g - 2 + n \geq 2$

- from $p \in \partial_1\Sigma$, shoot a geodesic η_p orthogonally to the boundary and stop it at the first intersection with itself or with $\partial\Sigma$
- apart from rare pathological cases, it determines an embedded pair of pants P_p with geodesic boundaries and bounding $\partial_1\Sigma$
- write $$1 = \frac{1}{\ell_\sigma(\partial_1\Sigma)} \sum_{[P] \in \mathcal{P}_\Sigma} \ell_\sigma\big(\{p \in \partial_1\Sigma \quad | \quad [P_p] = [P]\}\big)$$





III.1 McShane-type identities (hyperbolic)

$$\mathcal{P}_\Sigma^{\emptyset} = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{c} \partial_1 P = \partial_1 \Sigma \\ \partial_2 P = \partial_m \Sigma \end{array} \right\}$$

$$\mathcal{P}_\Sigma = \left(\bigcup_{m=2}^{n} \mathcal{P}_\Sigma^m \right) \cup \mathcal{P}_\Sigma^{\emptyset}$$

$$\mathcal{P}_\Sigma^{m} = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{c} \partial_1 P = \partial_1 \Sigma \\ \partial_{2,3} P \subset \mathring{\Sigma} \end{array} \right\}$$

$$B_{\mathrm{M}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big(F(L_1 + L_2 - \ell) + F(L_1 - L_2 - \ell) - F(-L_1 + L_2 - \ell) - F(-L_1 - L_2 - \ell)\big)$$

$$C_{\mathrm{M}}(L_1, \ell, \ell') = \frac{1}{L_1}\big(F(L_1 - \ell - \ell') - F(-L_1 - \ell - \ell')\big) \qquad \text{with } F(x) = 2\ln(1 + e^{x/2})$$

Theorem (Mirzakhani, 07) For $2g - 2 + n \geq 2$ and any $\sigma \in \mathcal{T}_\Sigma$

(a) $$1 = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^m} B_{\mathrm{M}}(\vec{\ell}_\sigma(\partial P)) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_\Sigma^{\emptyset}} C_{\mathrm{M}}(\vec{\ell}_\sigma(\partial P))$$

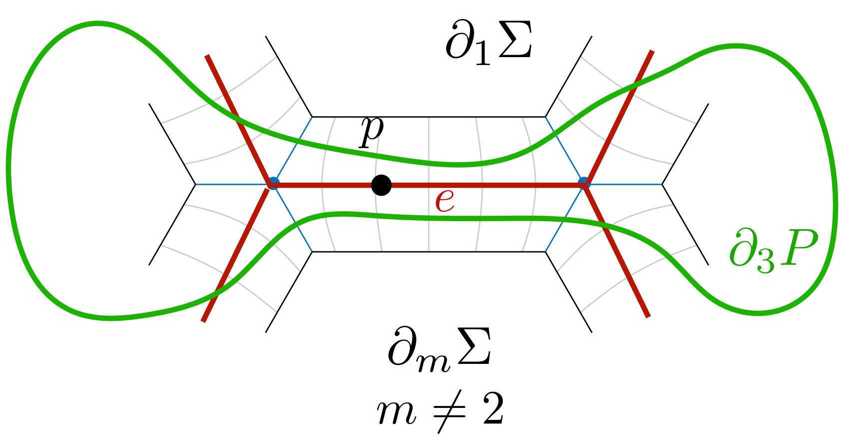
(b) Topological recursion for the WP volumes (using Wolpert's formula)

III.2 McShane-type identities (combinatorial)

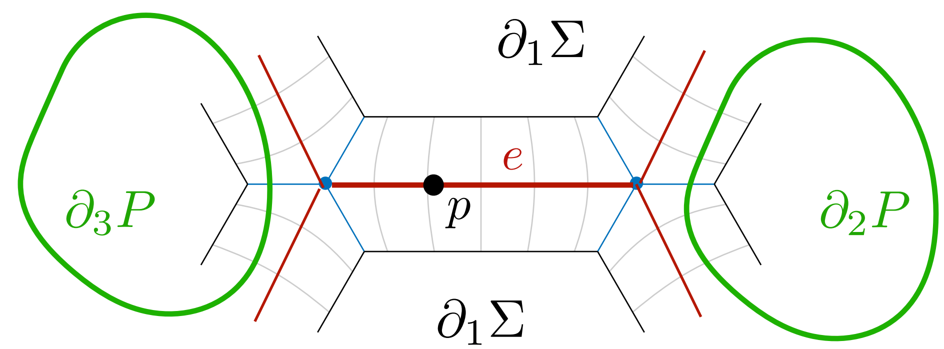
We can apply the same strategy in the combinatorial setting

Assume $2g-2+n \geq 2$ and a combinatorial structure $\mathbb{G}$ on Σ

- For each p, we have an associated $[P_p] \in \mathcal{P}_\Sigma$ depending only on the edge to which p belongs



$\ell_{\mathbb{G}}(e) = \ldots$



$\ell_{\mathbb{G}}(e) = \frac{1}{2}\big(L_1 - \ell_{\mathbb{G}}(\partial_2 P) - \ell_{\mathbb{G}}(\partial_3 P)\big)$

- Conversely, $[P] \in \mathcal{P}_\Sigma$ appears in this way (at most 3 times) iff $\ell_{\mathbb{G}}(\partial P \cap \partial\Sigma) \geq \ell_{\mathbb{G}}(\partial P \cap \mathring{\Sigma})$

III.2 McShane-type identities (combinatorial)

$$B_{\mathrm{K}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big([L_1 + L_2 - \ell]_+ + [L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+\big)$$
$$C_{\mathrm{K}}(L_1, \ell, \ell') = \frac{1}{L_1}[L_1 - \ell - \ell']_+$$

Proposition 8 **(ABCGLW 20)** For $2g - 2 + n \geq 2$ and any $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$

(a) $$1 = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^m} B_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big) + \frac{1}{2} \sum_{P \in \mathcal{P}_\Sigma^\emptyset} C_{\mathrm{K}}\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)$$

(b) Topological recursion for Kontsevich volumes (using Wolpert comb. formula)

⇝ geometric proof of the Virasoro part of Witten's conjecture

(a) and (b) can also be proved by flowing Mirzakhani's results from hyp. to comb. thanks to uniform control on lengths

Let M_Σ (resp. M'_Σ) be the set of (primitive) multicurves on Σ

and $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that $\varphi(\ell) \underset{\ell\to\infty}{=} O(\ell^{-\infty})$ and $\varphi(\ell) \underset{\ell\to 0}{=} O(\ell^{-2+\epsilon})$

We consider multiplicative statistics of lengths of multicurves

- hyperbolic world : $\sigma \in \mathcal{T}_\Sigma$ $$\Omega_{\mathrm{M}}[\varphi](\sigma) = \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_\sigma(\beta))$$
- combinatorial world : $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$ $$\Omega_{\mathrm{K}}[\varphi](\mathbb{G}) = \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_{\mathbb{G}}(\beta))$$

We can generalize Mirzakhani identity, to compute these functions by recursion

Using Wolpert formulas, this implies topological recursion for integrals over the moduli spaces

fits in a general theory : geometric recursion $\Longrightarrow$ topological recursion

(Andersen, B., Orantin, 17)

Let us define
$$B[f](L_1, L_2, \ell) = B(L_1, L_2, \ell) + f(\ell)$$
$$C[f](L_1, \ell, \ell') = C(L_1, \ell, \ell') + B(L_1, \ell, \ell')f(\ell) + B(L_1, \ell', \ell)f(\ell') + f(\ell)f(\ell')$$

Theorem 9 **(Andersen, B, Orantin 17)** For $2g - 2 + n \geq 2$

(a) $$\Omega_{\mathrm{M}}[\varphi](\sigma) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_{\Sigma}^{m}} B_{\mathrm{M}}[\varphi]\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\, \Omega_{\Sigma - P}(\sigma|_{\Sigma - P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_{\Sigma}^{\emptyset}} C_{\mathrm{M}}[\varphi]\big(\vec{\ell}_{\sigma}(\partial P)\big)\, \Omega_{\Sigma - P}[\varphi](\sigma|_{\Sigma - P})$$

(b) $$V\Omega_{g,n}^{\mathrm{M}}[\varphi](L) = \int_{\mathcal{M}_{g,n}(L)} \mathrm{d}\mu_{\mathrm{WP}}(\sigma)\, \Omega_{\Sigma}^{\mathrm{M}}[\varphi](\sigma)$$ exists and satisfies topological recursion

Theorem 9' **(ABCGLW 20)** For $2g - 2 + n \geq 2$

(a) $$\Omega_{\Sigma}^{\mathrm{K}}[\varphi](\mathbb{G}) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_{\Sigma}^{m}} B_{\mathrm{K}}[\varphi]\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\, \Omega_{\Sigma - P}^{\mathrm{K}}[\varphi](\mathbb{G}|_{\Sigma - P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_{\Sigma}^{\emptyset}} C_{\mathrm{K}}[\varphi]\big(\vec{\ell}_{\mathbb{G}}(\partial P)\big)\, \Omega_{\Sigma - P}^{\mathrm{K}}[\varphi](\mathbb{G}|_{\Sigma - P})$$

(b) $$V\Omega_{g,n}^{\mathrm{K}}[\varphi](L) = \int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} \mathrm{d}\mu_{\mathrm{K}}(\mathbb{G})\, \Omega_{\Sigma}^{\mathrm{K}}[\varphi](\mathbb{G})$$ exists and satisfies topological recursion

Idea of the proof of (a) same in hyperbolic or combinatorial worlds

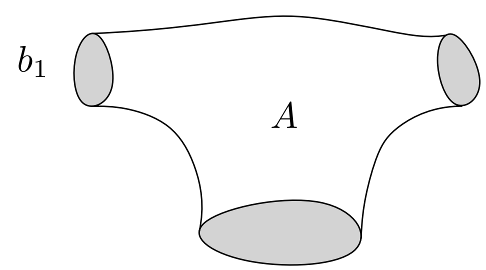
$$\Omega_{\mathrm{M}}[\varphi](\sigma) = \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_\sigma(\beta)) \cdot \mathbf{1}_{\Sigma - \gamma}(\sigma|_{\Sigma-\gamma})$$

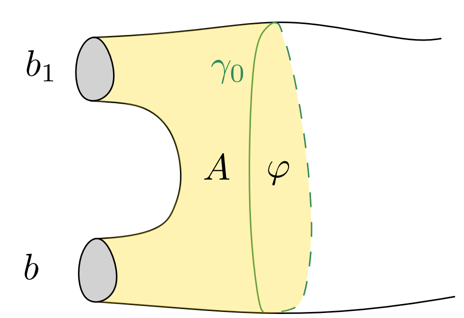
use Mirzakhani identity

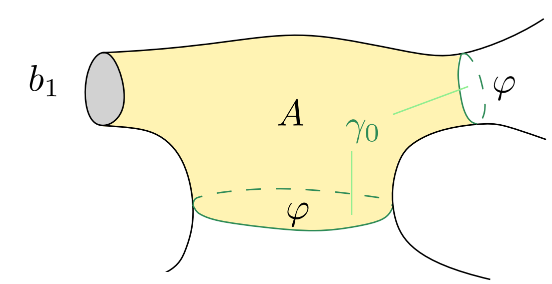
$$= \sum_{\gamma \in M'_\Sigma} \prod_{\beta \in \pi_0(\gamma)} \varphi(\ell_\sigma(\gamma)) \sum_{[P] \in \mathcal{P}_{\Sigma-\gamma}} X_{\mathrm{M},P}(\sigma|_{\Sigma - P})$$

$$= \sum_{[P] \in \mathcal{P}_\Sigma} \sum_{\gamma \in M'_{\Sigma - P}} \cdots$$

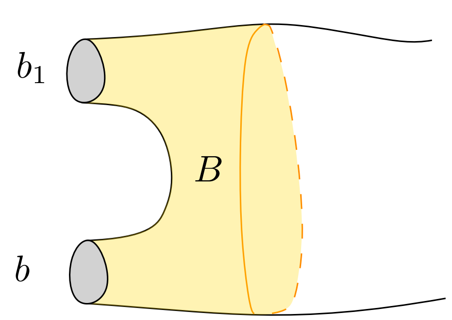
and collect the weights

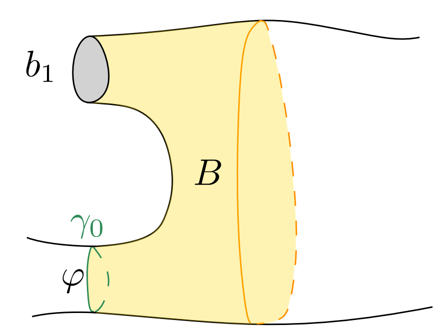


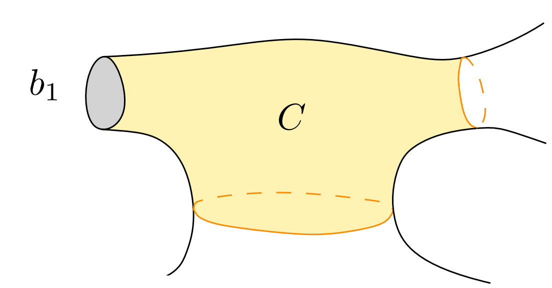




$$A = \Omega_{0,3} \equiv 1$$







IV

Thurston volume of unit balls

Let $\mathrm{MF}_\Sigma \subset \mathrm{MF}_\Sigma^\star$ be the set of measured foliations where $\partial\Sigma$ is a union of sing. leaves

It admits a piecewise linear integral structure and $\dim \mathrm{MF}_\Sigma = 6g-6+2n$

{Integral points of MF_Σ} = M_Σ = {multicurves}

Thurston measure of $A \subset \mathrm{MF}_\Sigma$ $\quad \mu_{\mathrm{Th}}(A) = \lim_{k\to\infty} \frac{|A \cap k^{-1}M_\Sigma|}{k^{6g-6+2n}}$ if exists

	Hyperbolic	Combinatorial
Length functions	$\mathcal{T}_\Sigma \times \mathrm{MF}_\Sigma \to \mathbb{R}_+$	$\mathcal{T}_\Sigma^{\mathrm{comb}} \times \mathrm{MF}_\Sigma \to \mathbb{R}_+$
Vol. of unit balls	$\mathcal{B}_\Sigma(\sigma) = \mu_{\mathrm{Th}}\big(\{\ell_\sigma \leq 1\}\big)$	$\mathcal{B}_\Sigma^{\mathrm{comb}}(\mathbb{G}) = \mu_{\mathrm{Th}}\big(\{\ell_{\mathbb{G}} \leq 1\}\big)$
Moments on Teichmüller	$V^s\mathcal{B}_{g,n}(L) := \int_{\mathcal{M}_{g,n}(L)} \mathrm{d}\mu_{\mathrm{WP}}(\sigma)\big(\mathcal{B}_\Sigma(\sigma)\big)^s$	$V^s\mathcal{B}_{g,n}^{\mathrm{comb}}(L) := \int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} \mathrm{d}\mu_{\mathrm{K}}(\mathbb{G})\big(\mathcal{B}_\Sigma(\mathbb{G})\big)^s$

Known results for punctured hyperbolic surfaces Σ

- $\mathcal{B}_\Sigma : \mathcal{T}_\Sigma \to \mathbb{R}_+$ is continuous, proper, and

$$c'_{g,n} \prod_{\substack{\gamma \in S_\Sigma \\ \ell_\sigma(\gamma) \leq \epsilon}} \frac{1}{\ell_\sigma(\gamma) \left| \ln(\ell_\sigma(\gamma)) \right|} \leq \mathcal{B}_\Sigma(\sigma) \leq c_{g,n} \prod_{\substack{\gamma \in S_\Sigma \\ \ell_\sigma(\gamma) \leq \epsilon}} \frac{1}{\ell_\sigma(\gamma)}$$

Mirzakhani (07)

$\Longrightarrow$ $V^s\mathcal{B}_{g,n}(0)$ is finite for $s < 2$ and infinite for $s > 2$

- Finer upper bound $\Longrightarrow$ $V^2\mathcal{B}_{g,n}(0)$ is finite

Arana-Herrera, Athreya (19)

- Relation to Masur-Veech volumes

$$\begin{array}{ccccc} \mathcal{Q}\mathcal{T}_\Sigma & \overset{\sim}{\dashrightarrow} & \mathrm{MF}_\Sigma \times \mathrm{MF}_\Sigma & \overset{\sim}{\dashleftarrow} & \mathcal{T}_\Sigma \times \mathrm{MF}_\Sigma \\ \mu_{\mathrm{MV}} & & \mu_{\mathrm{Th}} \otimes \mu_{\mathrm{Th}} & & \mu_{\mathrm{WP}} \otimes \mu_{\mathrm{Th}} \end{array}$$

Bonahon (96)
Mirzakhani (08)

$$\Longrightarrow \quad V^1\mathcal{B}_{g,n}(0) = \frac{\mu_{\mathrm{MV}}\big(\mathcal{Q}^1_{g,n}\big)}{2^{4g-2+n} \cdot (6g-6+2n) \cdot (4g-4+n)!}$$

Delecroix, Goujard, Zograf, Zorich (19)
Monin-Telpukhovskiy (19)
Arana-Herrera (19)

Open problem : compute explicitly $\mathcal{B}_\Sigma(\sigma)$ and $\big(V^s\mathcal{B}_{g,n}(L)\big)_{s \neq 1}$

- There are by now many ways to compute the Masur-Veech volumes
  (sums over stable graphs, 2 topological recursions, intersection theory on $\overline{\mathcal{M}}_{g,n}$)

  Mirzakhani (08)
  Delecroix, Goujard
  Zograf, Zorich (19)

  ABCDGLW (19)

  Chen, Möller, Sauvaget
  + B, Giacchetto, Lewanski (19)

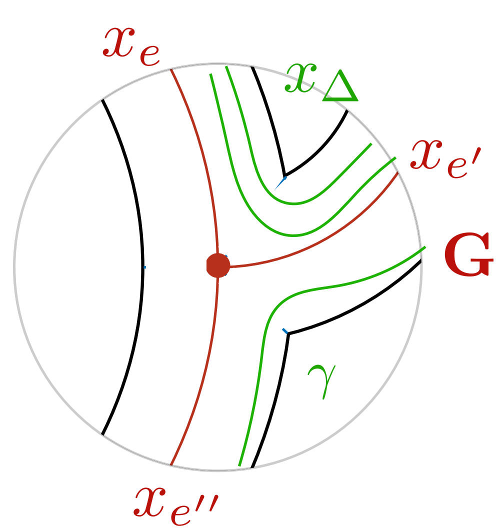
- $V^1 \mathcal{B}_{g,n}(L)$ is independent of $L \in \mathbb{R}^n_{\geq 0}$ ABCDGLW (19)

Problem : compute explicitly $\mathcal{B}_\Sigma(\sigma)$ and $\big(V^s \mathcal{B}_{g,n}(L)\big)_{s \neq 1}$?

The combinatorial setting is easier as one can make explicit computations

$M_\Sigma^\bullet =$ {multicurves, possibly including components homotopic to boundaries}

Let $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$ and assume the underlying ribbon graph $\mathbf{G}$ is trivalent



$\ell_{\mathbb{G}}(\gamma) = \sum_{e \in E_G} x_e \, \ell_{\mathbb{G}}(e)$

$x_e =$ # times γ travels along e $\qquad x_\Delta = \dfrac{x_e + x_{e'} - x_{e''}}{2}$

$$M_\Sigma^\bullet \xrightarrow{\sim} Z_{\mathbf{G}}^\bullet = \{x \in \mathbb{N}^{E_G} \quad | \quad \forall \Delta \quad x_\Delta \in \mathbb{N}\}$$

$$M_\Sigma \xrightarrow{\sim} Z_{\mathbf{G}} := \{x \in Z_{\mathbf{G}}^\bullet \quad | \quad \forall i \quad \min_{\Delta \circlearrowleft \partial_i \mathbf{G}} x_\Delta = 0\}$$

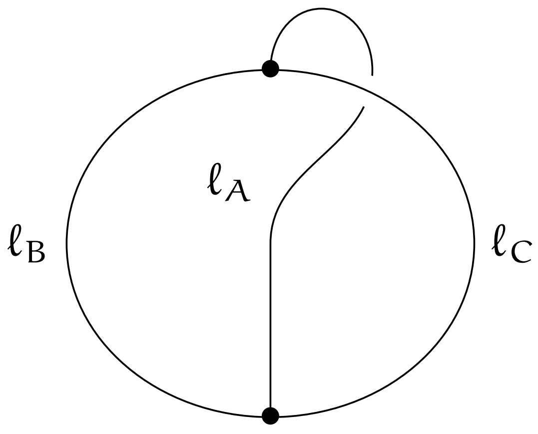
Lemma 10 $\mathcal{B}_\Sigma^{\mathrm{comb}}(\mathbb{G})$ is the euclidean volume of a union of polytopes in $\mathbb{R}^{6g-6+2n}$

$$= \sum_i \frac{1}{d_i} \frac{1}{\prod_{r \in R_{G,i}} \ell_{\mathbb{G}}(r)}$$

where $R_{G,i}$ is a set of (6g - 6 + 2n) simple cycles and dumbbells, and $d_i \in \mathbb{N}^*$

Example

$(g,n) = (1,1)$



$L = 2(\ell_A + \ell_B + \ell_C)$

generically $\mathbb{Z}_3 \times \mathbb{Z}_2$- symmetry

$$\mathcal{B}_{\Sigma}^{\mathrm{comb}}(\ell_A, \ell_B, \ell_C) = \frac{1}{2}\,\frac{1}{(\ell_A + \ell_B)(\ell_B + \ell_C)} + \mathrm{cyc.}$$

$$= \frac{L/2}{(\ell_A + \ell_B)(\ell_B + \ell_C)(\ell_C + \ell_A)}$$

$$V^s\mathcal{B}_{1,1}^{\mathrm{comb}}(L) = \frac{(L/2)^{s-1}}{6} \int_{0 \le a+b \le 1} \frac{\mathrm{d}a\mathrm{d}b}{\big((a+b)(1-a)(1-b)\big)^s}$$

in particular $V^1\mathcal{B}_{1,1}^{\mathrm{comb}}(L) = \frac{1}{16} \cdot \frac{2\pi^2}{3} = V^1\mathcal{B}_{1,1}(L)$

We can deduce that $V^s\mathcal{B}_{1,1}^{\mathrm{comb}}(L)$ is finite iff $s < 2$, and has a simple pole at $s = 2$

IV.3 Thurston volume of unit balls — Combinatorial case

Lemma 11 $V^1\mathscr{B}_{g,n}(L) = V^1\mathscr{B}^{\mathrm{comb}}_{g,n}(L)$ is independent of $L \in \mathbb{R}^n_{\geq 0}$

(proof : because both can be computed)

so Masur-Veech volumes can be approached as well from combinatorial geometry (bypassing horocyclic foliation and hyp. geodesic dynamics)

In general, there is less integrability than in the hyperbolic case (absence of collar lemma)

Theorem 12 **(B, Charbonnier, Delecroix, Giacchetto, Wheeler, to appear)**

$V^s\mathscr{B}^{\mathrm{comb}}_{g,n}(L)$ is finite

iff $s < s^*_{g,n} \leq 2$

<table>
<tr><th>g/n</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>$\geq$ 6</th></tr>
<tr><td>0</td><td colspan="2"></td><td>∞</td><td>2</td><td>2</td><td>$\frac{4}{3} + \frac{1}{2(\lfloor n/2 \rfloor - 2)}$</td></tr>
<tr><td>1</td><td>2</td><td colspan="5">$\frac{4}{3}$</td></tr>
<tr><td>2</td><td>$\frac{4}{3}$</td><td colspan="5">$1 + \frac{1}{3(2g-1)}$</td></tr>
<tr><td>$\geq$ 3</td><td colspan="6">$1 + \frac{1}{3(2g-3)}$</td></tr>
</table>

IV.4 Thurston volume of unit balls — Comparison hyp./comb.

$$
\begin{array}{ccccc}
\sigma & \mathcal{T}_\Sigma(L) & \xrightarrow{\ \mathrm{sp}\ } & \mathcal{T}_\Sigma^{\mathrm{comb}}(L) & \mathbb{G} \\
\downarrow & \downarrow \Phi_\beta & \swarrow \rho_\beta & \downarrow & \downarrow \\
\sigma^\beta & \mathcal{T}_\Sigma(\beta L) & \xleftarrow{\ \mathrm{sp}^{-1}\ } & \mathcal{T}_\Sigma^{\mathrm{comb}}(\beta L) & \beta\mathbb{G}
\end{array}
$$

Jacobian

$$J_\beta := \frac{1}{\beta^{6g-6+2n}} \frac{\rho_\beta^* \mathrm{d}\mu_{\mathrm{WP}}}{\mathrm{d}\mu_{\mathrm{K}}}$$

- By Lemma 5 $\lim_{\beta\to\infty} \beta^{6g-6+2n} \rho_\beta^* \mathcal{B}_\Sigma = \mathcal{B}_\Sigma^{\mathrm{comb}}$ uniform cv. on thick parts of $\mathcal{T}_\Sigma^{\mathrm{comb}}$

- By Mondello (09) $\lim_{\beta\to\infty} J_\beta = 1$

Fatou lemma $\Longrightarrow$ $V^s\mathcal{B}_{g,n}^{\mathrm{comb}}(L) \leq \liminf_{\beta\to\infty} \frac{V^s\mathcal{B}_{g,n}(\beta L)}{\beta^{(6g-6+2n)(s-1)}}$

- For $s \geq s_{g,n}^*$, LHS infinite $\Rightarrow$ anomalous scaling of $V^s\mathcal{B}_{g,n}(L)$ for large length

- By Lemma 10, for $s = 1$, both sides are equal (independent of L thus β)

Miss a uniform 'integrable' bound on J_β to study equality for $s < s_{g,n}^*$

Thank you for your attention !

A. Giacchetto

based on

Topological recursion for Masur-Veech volumes
with J.E. Andersen, S. Charbonnier, V. Delecroix, A. Giacchetto, D. Lewanski, C. Wheeler
math.GT/1905.10352

On the Kontsevich geometry of the combinatorial Teichmüller space
with J.E. Andersen, S. Charbonnier, A. Giacchetto, D. Lewanski, C. Wheeler
to appear

Around the combinatorial unit ball of measured foliations on bordered surfaces
with S. Charbonnier, V. Delecroix, A. Giacchetto, C. Wheeler
to appear