A thought for Vaughan Jones

From Dror Bar-Natan's image gallery

Renew Quantum internal seminar@online

Classifying Airy structures from $W(\mathfrak{gl}_r)$ algebras

based on joint works with
Andersen, Chekhov, Orantin : 1703.03307
Bouchard, Chidambaram, Creutzig, Noshchenko: 1812.08738
Kramer, Schüler : to appear

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Problem

What is the minimal framework needed to define topological recursion ?

The original work of Eynard emphasized the role of "spectral curves"

These are complex curves with extra data, and TR builds from it a sequence of multidifferentials $\{\omega_{g,n} \ : \ g \in \mathbb{N}, \ n \in \mathbb{N}^*\}$ by induction on 2g - 2 + n

Problem

What is the minimal framework needed to define topological recursion ?

TR solves many problems in enumerative geometry

The enumerative information is stored in the periods of $\omega_{g,n}$

Bouchard-Klemm-Mariño-Pasquetti (07) proposed to see TR as
the definition of the B-model amplitudes associated to the spectral curve

The link to enumerative geometry (for them, GW of toric CY3) is an instance of mirror symmetry

Problem

What is the minimal framework needed to define topological recursion ?

Eynard and Orantin's insights (07) were

- promoting TR to an intrinsic construction from the geometry of spectral curves (independently of their origin from matrix models or mirrors of CY3, …)
- stressing its properties in greater generality (link to special geometry, holomorphic anomaly equations, symplectic invariance, …)

This perspective led to the discovery of many new applications of TR

(Weil-Petersson volumes, intersection theory on $\overline{\mathcal{M}}_{g,n}$, CohFTs, Hitchin systems, WKB expansions, knot theory, special geometry, …)

Problem

What is the minimal framework needed to define topological recursion ?

As of now, there are no satisfactory set of assumptions (nice, minimal, general enough) specifying for which spectral curves TR should be definable

However, we expect that having it would

- enlighten the profound algebraic nature of the ‘invariants’ that TR constructs
- explain its properties (notably : symplectic invariance) by relating it to deep (?) results in algebraic geometry
- relate to a classification of 2d topological field theories

Definition

A spectral curve is a quadruple S = (C,x,y, $\omega_{0,2}$) where

C is a complex curve

x meromorphic function $\rightsquigarrow$ $\mathfrak{a}$ = {zeroes of dx}

y meromorphic function $\rightsquigarrow$ $\omega_{0,1} = y\mathrm{d}x$

$\omega_{0,2} \in H^0(K_C^{\boxtimes 2}(2\Delta), C^2)^{\mathfrak{S}_2}$ has biresidue 1 on the diagonal Δ

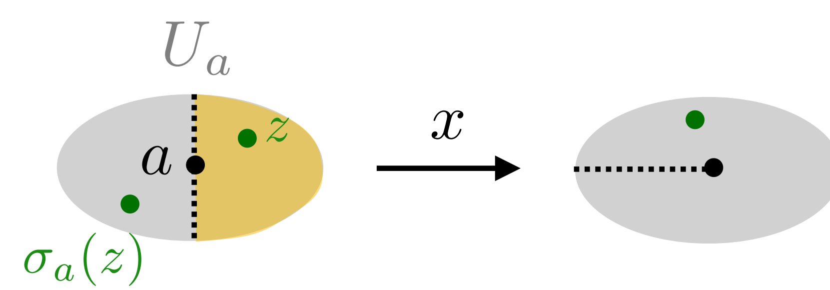
The output of TR will then be, for each $g \in \mathbb{N},\ n \in \mathbb{N}^*$

$\omega_{g,n} \in H^0\big(K_C(*\mathfrak{a})^{\boxtimes n}, C^n\big)^{\mathfrak{S}_n}$

Original setting (Eynard-Orantin, 07)

Assumptions

C is smooth, $\mathfrak{a}$ is finite and for each $a \in \mathfrak{a}$



$\mathrm{d}y(a) \neq 0$

a is a simple zero of dx

$\rightsquigarrow$ in a small neighborhood U_a of a $\quad x^{-1}(x(z)) \cap U_a = \{z, \sigma_a(z)\}$

Definition By induction on 2g - 2 + n > 0

$$\omega_{g,n}(z_1, \underbrace{z_2, \ldots, z_n}_{I}) =$$

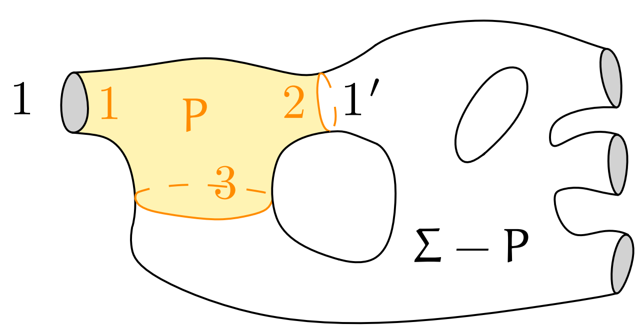
$$\sum_{a \in \mathfrak{a}} \operatorname*{Res}_{z=a} \frac{\int_a^z \omega_{0,2}(\cdot, z_1)}{\big(y(\sigma_a(z)) - y(z)\big)\mathrm{d}x(z)} \Bigg(\omega_{g-1,n+1}(z, \sigma_a(z), I) + \sum_{\substack{h+h'=g \\ J \sqcup J' = I}}^{\text{no } \omega_{0,1}} \omega_{h,1+|J|}(z, J)\,\omega_{h',1+|J'|}(\sigma_a(z), J')\Bigg)$$

$$\omega_{g,n}(z_1, \underbrace{z_2, \ldots, z_n}_{I}) =$$

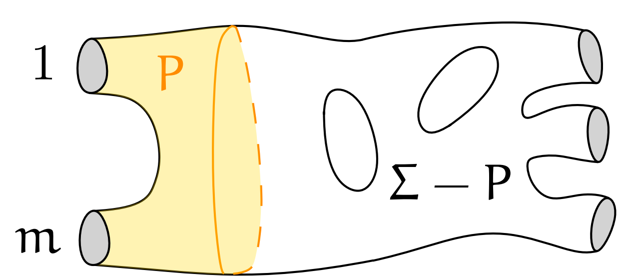
$$\sum_{a \in \mathfrak{a}} \operatorname*{Res}_{z=a} \frac{- \int_a^z \omega_{0,2}(\cdot, z_1)}{\big(y(\sigma_a(z)) - y(z)\big)\mathrm{d}x(z)} \left(\omega_{g-1,n+1}(z, \sigma_a(z), I) + \sum_{\substack{h+h'=g \\ J \sqcup J' = I}}^{\text{no } \omega_{0,1}} \omega_{h,1+|J|}(z, J)\omega_{h',1+|J'|}(\sigma_a(z), J') \right)$$

Symmetric in $z_1, \ldots, z_n$ although the definition is not

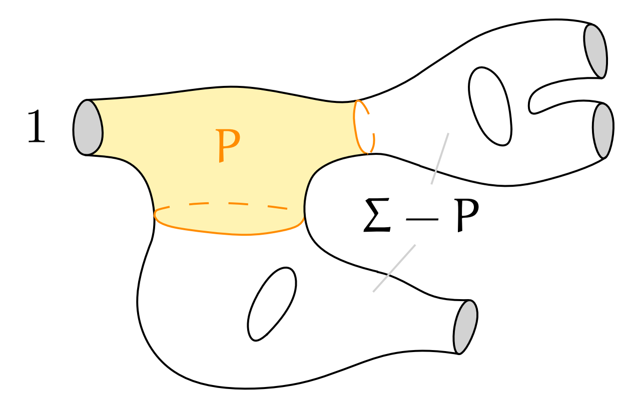
Terms are in 1:1 correspondence with diffeo class. of embedded pairs of pants $P \hookrightarrow \Sigma_{g,n}$ such that $\partial_1 P = \partial_1 \Sigma_{g,n}$ and $\chi(\Sigma_{g,n} - P) < 0$



$\omega_{g-1,n+1}$



$\omega_{0,2}\,\omega_{g,n-1}$



$\omega_{h,1+|J|}\,\omega_{h',1+|J'|}$

Assumptions C is smooth, $\mathfrak{a}$ is finite

(EO 07) For each $a \in \mathfrak{a}$, $\mathrm{d}y(a) \neq 0$

a is a simple zero of dx

Other behavior for y ?

Definition only depends on local information near as

Insensitive to the invariant part of y under σ_a

If y = 0 near a : ill-defined.

Otherwise, near a : $y \sim c_a \cdot (x - x(a))^{s_a/2-1} \mod \mathbb{C}(x)$

$s_a \leq -1$: $\omega_{g,n} = 0$ for 2g - 2 + n > 0

$s_a = 1$: application in the works of Chekhov, Do, Norbury, …

$s_a = 3$: majority of applications

$s_a \geq 5$: $\omega_{0,3}$ not symmetric

Can one find a good definition of TR for more general spectral curves ?

A good definition of TR means :

- $\omega_{g,n}$ defined by recursion on 2g - 2 + n > 0
- Terms are in 1:1 correspondence with diffeo. class of embedded stable surfaces $\Sigma' \hookrightarrow \Sigma_{g,n}$ such that $\partial_1 \Sigma' = \partial_1 \Sigma_{g,n}$ and $|\chi(\Sigma')| < 2g - 2 + n$
- it reduces to EO definition when C is smooth, dx has simple zeroes at which dy ≠ 0
- $\omega_{g,n}(z_1, \ldots, z_n)$ is symmetric in $z_1, \ldots, z_n$

1. Higher order ramification points

2. Singular curves

3. Airy structures from W-algebras :
correspondence with TR and investigation of symmetry

1. Higher order ramification points

1. Higher order ramification points

Bouchard, Hutchinson, Loliencar, Meiers, Rupert (12) proposed a definition for higher order ramifications

Assumptions

C is smooth, $\mathfrak{a}$ is finite

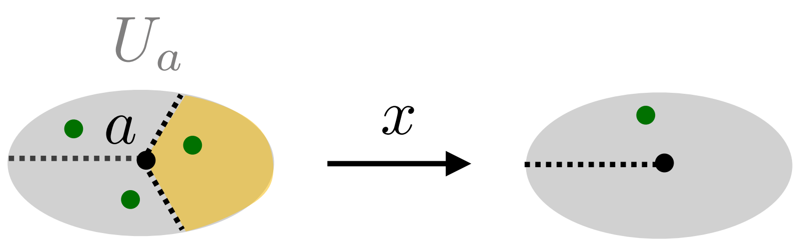
For each $a \in \mathfrak{a}$, $\mathrm{d}y(a) \neq 0$

a is a simple zero of order $r_a - 1$ of dx

$r_a \geq 2$

$$\mathfrak{f}_a(z) := x^{-1}(x(z)) \cap U_a = \{z, \sigma_a(z), \ldots, \sigma_a^{r_a - 1}(z)\}$$

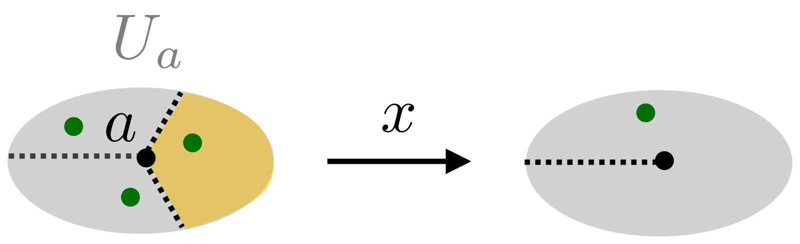
$$\mathfrak{f}'_a(z) := \mathfrak{f}_a(z) \setminus \{z\}$$



1. Higher order ramification points

$$\mathfrak{f}_a(z) := x^{-1}(x(z)) \cap U_a = \{z, \sigma_a(z), \dots, \sigma_a^{r_a-1}(z)\}$$

$$\mathfrak{f}'_a(z) := \mathfrak{f}_a(z) \setminus \{z\}$$



Recursion kernel $$K_a^{(m)}(z_1, z, Z) = -\frac{\int_a^z \omega_{0,2}(z_1, \cdot)}{\prod_{z' \in Z} \big(y(z') - y(z)\big) \mathrm{d}x(z)}$$

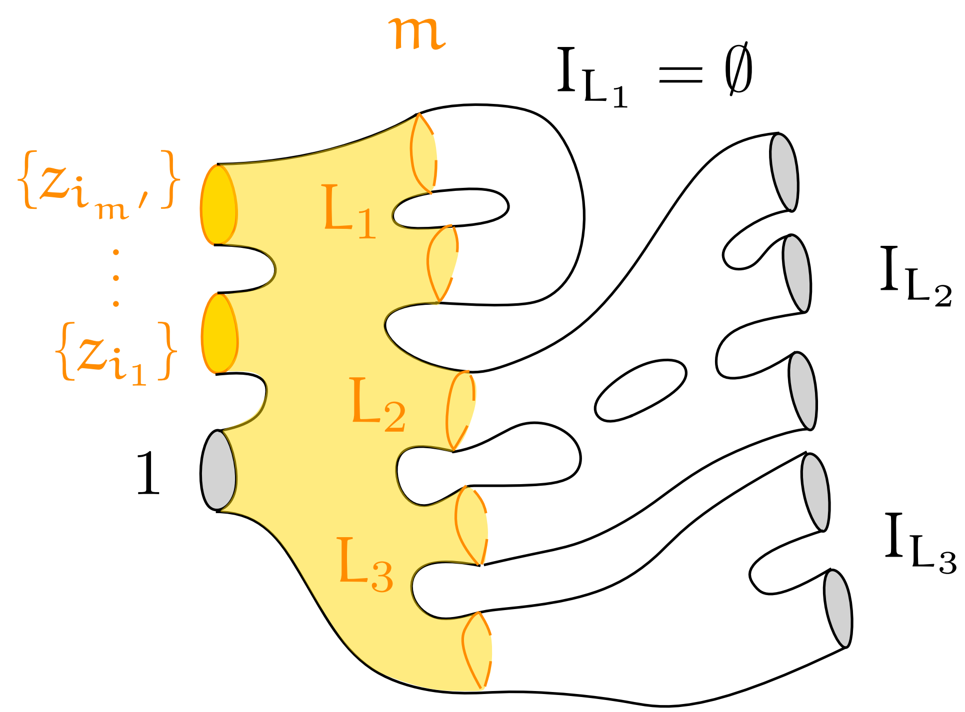
Known from induction : $$\Omega_{g,m,n}(Z; I) := \sum_{\substack{\mathbf{L} \vdash Z \\ \sqcup_{L \in \mathbf{L}} I_L = I \\ m + \sum_L (g_L - 1) = g}}^{\text{no } \omega_{0,1}} \prod_{L \in \mathbf{L}} \omega_{g_L, |L| + |I_L|}(Z, I_L)$$

Recursion formula :

$$\omega_{g,n}(z_1, \underbrace{z_2, \dots, z_n}_{I}) = \sum_{a \in \mathfrak{a}} \operatorname*{Res}_{z=a} \left(\sum_{Z \subseteq \mathfrak{f}'_a(z)} K_a^{(|Z|+1)}(z_1, z, Z) \Omega_{g,|Z|,I}(z, Z; I) \right)$$

1. Higher order ramification points

Terms are in 1:1 correspondence with $[\Sigma'_{0,1+m+m'} \hookrightarrow \Sigma_{g,n}]$



To evaluate their contribution :

Label the by the elements of $Z \subseteq \mathfrak{f}'_a(z) = \{\sigma_a(z), \ldots, \sigma_a^{r_a - 1}(z)\}$

For each stable connected component that remains after excision

with labeled by $\emptyset \neq L \subseteq Z$ labeled by $I_L \subseteq I = \{z_2, \ldots, z_n\}$ the weight is $\omega_{g_L, |L|+|I_L|}(L, I_L)$

A left corresponds by convention to $L = \{z'\}$, $I_L = \{z_i\}$, $g_L = 0$

and its weight is $\omega_{0,2}(z', z_i)$

1. Higher order ramification points

Assumptions C is smooth, $\mathfrak{a}$ is finite

For each $a \in \mathfrak{a}$, $\mathrm{d}y(a) \neq 0$

a is a simple zero of order $r_a - 1$ of dx $r_a \geq 2$

Bouchard-Eynard (13) give an argument to prove symmetry of $\omega_{g,n}$ in that case by deforming to a regular curve. It applies when

$y \sim c_a \cdot (x - x(a))^{s_a/r_a - 1} \bmod \mathbb{C}(x)$ with $s_a = r_a \pm 1$

Theorem 1 (Bouchard, B., Chidambaram, Creutzig, Noshchenko 18)

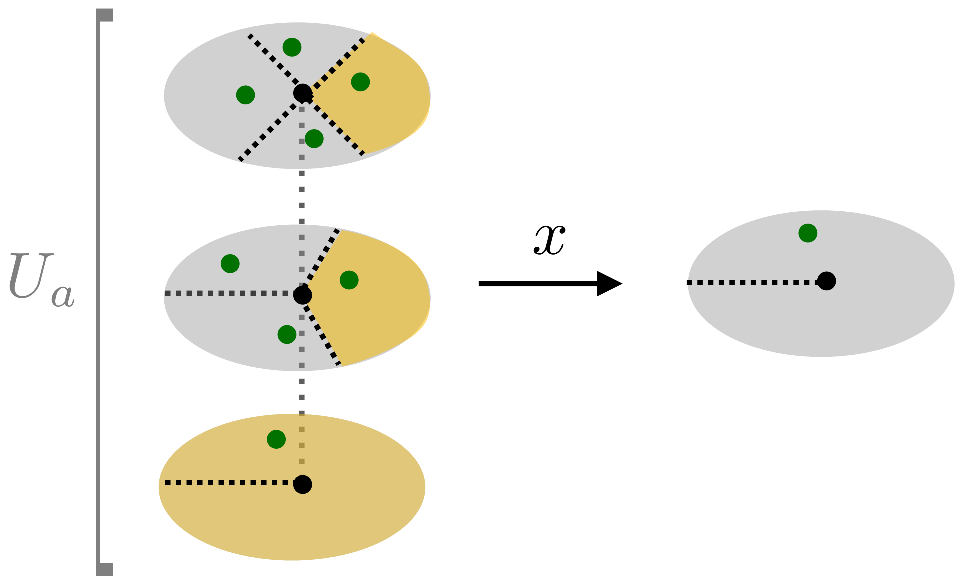
$\omega_{g,n}$ is symmetric if and only if

$s_a \in \{1, \ldots, r_a + 1\}$ and $r_a = \pm 1 \bmod s_a$

or $s_a < 0$ (in which case contributions from $\operatorname*{Res}_{z=a}$ vanish)

2. Singular spectral curves

Let C is be singular curve (with zeroes of dx at singular points)



Locally around each $a \in \mathfrak{a}$, x admit a ramification profile $(r_\mu)_{\mu=1}^{d_a}$

Here $(r_1, r_2, r_3) = (4, 3, 1)$

Examples nodal curve, with a zero of dx at the node

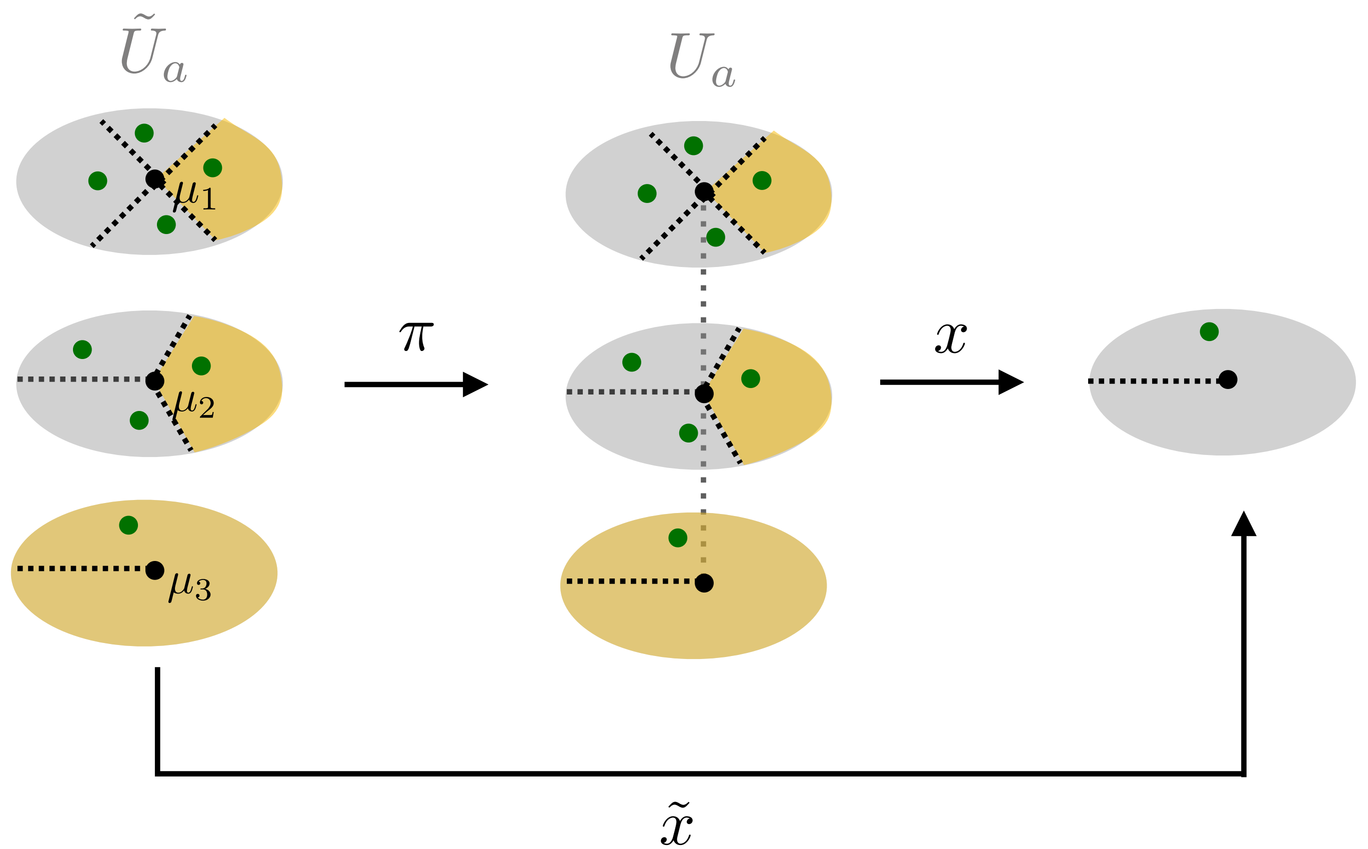
reducible curve, such as $(y^2 - x)^2 = 0$

$$\prod_{\mu=1}^{d} (y^{r_\mu} - x^{s_\mu - r_\mu}) = 0$$

2. Singular spectral curves

Let C is singular curve (with zeroes of dx at singular points)

In a normalisation $\pi : \tilde{C} \to C$, let $\tilde{\mathfrak{a}}_a \subset \tilde{U}_a$ be the set of zeroes of $\mathrm{d}\tilde{x}$



For $\mu \in \tilde{\mathfrak{a}}_a$, we denote $r_\mu - 1$ the order of the zero of $\mathrm{d}\tilde{x}$ at μ $(r_\mu \geq 1)$

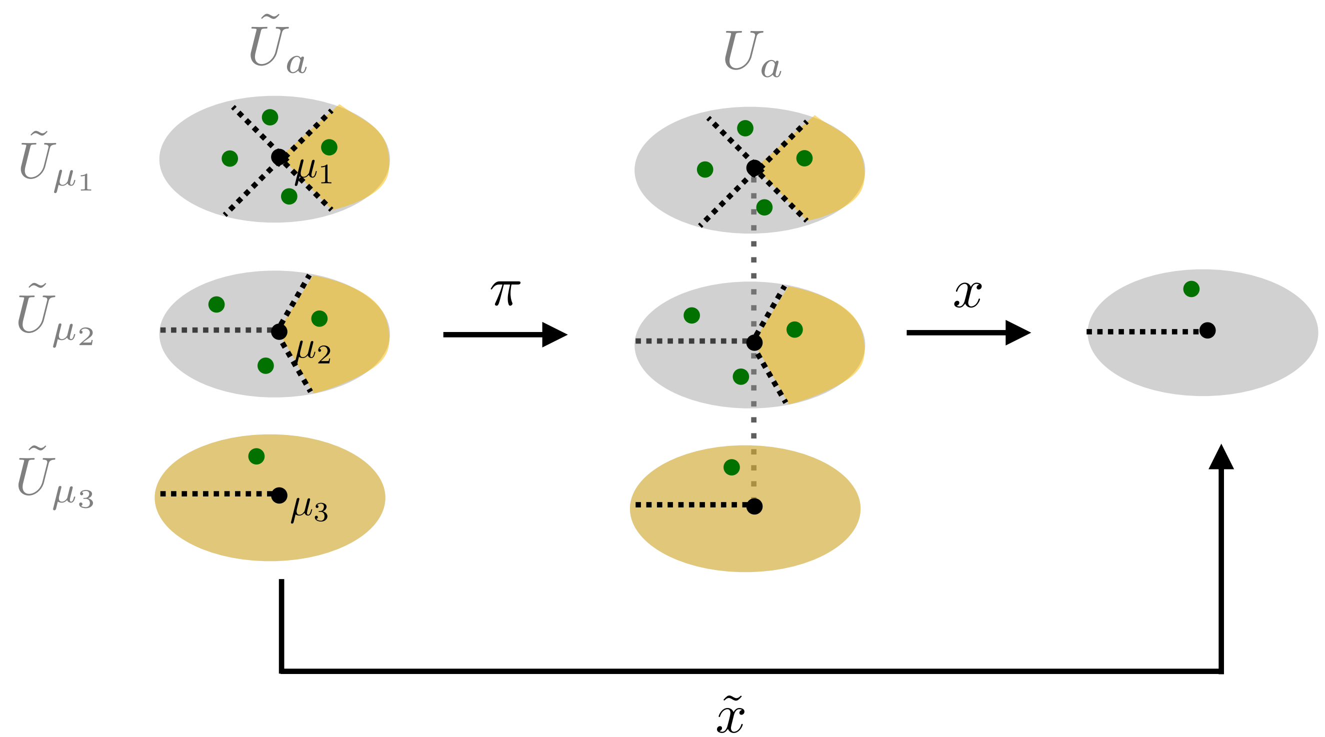
To define TR, we will rather work on the normalisation

Definition A singular spectral curve is the data of $(\pi : \tilde{C} \to C, x, y, \omega_{0,2})$
where $\pi : \tilde{C} \to C$ is a normalisation of complex curves
x,y are meromorphic functions on C
$\omega_{0,2} \in H^0(K_{\tilde{C}}^{\boxtimes 2}(2\Delta), \tilde{C}^2)^{\mathfrak{S}_2}$ has biresidue 1 on the diagonal Δ

$\rightsquigarrow$ $\tilde{x} = x \circ \pi$, $\tilde{y} = y \circ \pi$ are meromorphic functions on $\tilde{C}$

Each zero $a \in \mathfrak{a}$ of $\mathrm{d}x$ splits into a set $\tilde{\mathfrak{a}}_a = \pi^{-1}(a)$ of zeroes of $\mathrm{d}\tilde{x}$

For $\mu \in \tilde{\mathfrak{a}}_a$, we denote $r_\mu - 1$ the order of the zero of $\mathrm{d}\tilde{x}$ at μ $(r_\mu \geq 1)$



It is natural to propose the following definition of TR

$$\mathfrak{f}_a(z) := \tilde{x}^{-1}(x(z)) \cap \tilde{U}_a$$

$$\mathfrak{f}'_a(z) := \mathfrak{f}_a(z) \setminus \{z\}$$

Recursion kernel $$K_\mu^{(m)}(z_1, z, Z) = -\frac{\int_\mu^z \omega_{0,2}(z_1, \cdot)}{\prod_{z' \in Z} \big(\tilde{y}(z') - \tilde{y}(z)\big) \mathrm{d}\tilde{x}(z)}$$

Known from induction : $$\Omega_{g,m,n}(Z; I) := \sum_{\substack{\mathbf{L} \vdash Z \\ \sqcup_{L \in \mathbf{L}} I_L = I \\ m + \sum_L (g_L - 1) = g}}^{\text{no } \omega_{0,1}} \prod_{L \in \mathbf{L}} \omega_{g_L, |L| + |I_L|}(Z, I_L)$$

Recursion formula :

$$\omega_{g,n}(z_1, \underbrace{z_2, \dots, z_n}_{I}) = \sum_{a \in \mathfrak{a}} \sum_{\mu \in \tilde{\mathfrak{a}}_a} \operatorname*{Res}_{z=\mu} \left(\sum_{Z \subseteq \mathfrak{f}'_a(z)} K_\mu^{(|Z|+1)}(z_1, z, Z) \Omega_{g,|Z|,I}(z, Z; I) \right)$$

Same structure as before (using $\tilde{C}$), but the fiber $\mathfrak{f}_a(z)$ is larger

Each zero $a \in \mathfrak{a}$ of $\mathrm{d}x$ splits into a set $\tilde{\mathfrak{a}}_a = \pi^{-1}(a)$ of zeroes of $\mathrm{d}\tilde{x}$

For $\mu \in \tilde{\mathfrak{a}}_a$, we denote $r_\mu - 1$ the order of the zero of $\mathrm{d}\tilde{x}$ at μ $(r_\mu \geq 1)$

Near μ we have $\tilde{y} \sim c_\mu \cdot (\tilde{x} - \tilde{x}(\mu))^{s_\mu / r_\mu - 1} \bmod \mathbb{C}(\tilde{x})$ for some $s_\mu \in \mathbb{Z} \cup \{\infty\}$

We can always identify $\tilde{\mathfrak{a}}_a \simeq \{1, \ldots, d_a\}$ so that $\frac{s_{\mu_1}}{r_{\mu_1}} \leq \cdots \leq \frac{s_{\mu_{d_a}}}{r_{\mu_{d_a}}}$

Theorem 2 (B., Kramer, Schüler, 20)

For each $a \in \mathfrak{a}$, assume that

1. For each $\mu \in \tilde{\mathfrak{a}}_a$, we have $s_\mu \in \{1, \ldots, r_\mu + 1\}$
   except for $s_{\mu_{d_a}}$ which could also be ∞ if $d_a \geq 2$
2. If $d_a = 1$ (a is smooth), then $r_a = \pm 1 \bmod s_a$
3. If $d_a \geq 2$, then $r_{\mu_1} = -1 \bmod s_{\mu_1}$ & $s_{\mu_2} = \cdots = s_{\mu_{d_a - 1}} = 1$ & $r_{\mu_{d_a}} = 1 \bmod s_{\mu_{d_a}}$
4. $c_\mu^{r_\mu} \neq c_\nu^{r_\nu}$ whenever $r_\mu s_\nu = s_\mu r_\nu$ for distinct $\mu, \nu \in \tilde{\mathfrak{a}}_a$

Then $\omega_{g,n}$ is symmetric

Examples : fitting the assumptions

$y(y^r - x) = 0$

$(xy^2 - 1)(y^2 - x) = 0$

not fitting the assumptions

$(y^2 - x)^2 = 0$, $y^2 = 0$, reducible curves

$\rightsquigarrow$ recursion kernel ill-defined

$(xy^2 - 1)(x(y-1)^2 - 1) = 0$

$\rightsquigarrow$ $\omega_{1,2}(z_1, z_2)$ non-symmetric

Theorem 2 (B., Kramer, Schüler, 20)

For each $a \in \mathfrak{a}$, assume that

1. For each $\mu \in \tilde{\mathfrak{a}}_a$, we have $s_\mu \in \{1, \ldots, r_\mu + 1\}$ except for $s_{\mu_{d_a}}$ which could also be ∞ if $d_a \geq 2$
2. If $d_a = 1$ (a is smooth), then $r_a = \pm 1 \text{ mod } s_a$
3. If $d_a \geq 2$, then $r_{\mu_1} = -1 \text{ mod } s_{\mu_1}$ & $s_{\mu_2} = \cdots = s_{\mu_{d_a - 1}} = 1$ & $r_{\mu_{d_a}} = 1 \text{ mod } s_{\mu_{d_a}}$
4. $c_\mu^{r_\mu} \neq c_\nu^{r_\nu}$ whenever $r_\mu s_\nu = s_\mu r_\nu$ for distinct $\mu, \nu \in \tilde{\mathfrak{a}}_a$

Then $\omega_{g,n}$ is symmetric

We can get necessary conditions on $(r_\mu, s_\mu, c_\mu)_\mu$ to get symmetry by examining low (g,n)

For symmetry of $\omega_{0,3}$ and $\omega_{0,4}$, they are weaker than the sufficient conditions of Thm 2.

Theorem 3 (B., Kramer, Schüler, 20)

If the proposed recursion yields symmetric $\omega_{0,3}$ and $\omega_{0,4}$ for generic values of $(c_\mu)_\mu$ and $\gcd(r_\mu, s_\mu) = 1$ for all μ

Then for all $a \in \mathfrak{a}$ we must have 1., 2. and

3′. If $d_a \geq 2$, then $r_{\mu_1} = -1 \bmod s_{\mu_1}$ & $s_{\mu_2}, \ldots, s_{\mu_{d_a - 1}} \in \{1, 2\}$ & $r_{\mu_{d_a}} = 1 \bmod s_{\mu_{d_a}}$

I believe the conditions of Thm 2. are optimal for generic $(c_\mu)_\mu$

This puts constraints on the (naive) definition of deformation theory
(no Frobenius mfd structure ?)

Perhaps TR would still have a good definition in those pathological cases,
but it would have to be different.

There are external motivations to look for such a thing.

Eg : can one reconstruct from some TR the WKB expansion of solutions
of an ODE whose characteristic variety is $(y-1)^r = 0$?

(example : Picard-Fuchs equation for compact CY3 — having a point in moduli with
maximal unipotent monodromy)

2. Singular spectral curves

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3. Airy structures from W-algebras

3. Airy structures and TR

Airy structures (Kontsevich-Soibelman, 17) provide the minimal algebraic framework in which topological recursion can be defined (not necessarily based on spectral curves)

Let V = complex vector space, with a basis of linear coordinates $(x_i)_{i\in\mathcal{I}}$

$\mathcal{D}_{V,\hbar} = \mathbb{C}[\hbar, (\hbar\partial_{x_i})_i, (x_i)_i]$ graded algebra of differential operators on V

$\deg x_i = 1,\ \deg \hbar = 2$

Definition An Airy structure is a family $(H_i)_{i\in\mathcal{I}}$ of elements of $\mathcal{D}_{V,\hbar}$ satisfying

- degree 1 condition : $H_i = \hbar\partial_{x_i} + O(2)$
- Lie ideal condition : $\hbar^{-1}[\mathcal{A},\mathcal{A}] \subseteq \mathcal{D}_{V,\hbar}\cdot\mathcal{A}$ with $\mathcal{A} := \underset{k\in\mathcal{I}}{\mathrm{span}}(H_k)$

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Main property (KS 17) There exists a unique formal function on V

$$F = \sum_{\substack{g\geq 0,\ n>0 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} F_{g,n} \quad \text{with} \quad F_{g,n} \in \mathrm{Sym}(V^*)^{\otimes n}$$

such that $\forall i \in \mathcal{I},\ H_i e^F = 0$

$F_{g,n}$ computed by a recursion on 2g - 2 + n > 0

Terms are in 1:1 correspondence with diffeo. class of embedded stable surfaces $\Sigma' \hookrightarrow \Sigma_{g,n}$ such that $\partial_1\Sigma' = \partial_1\Sigma_{g,n}$ and $|\chi(\Sigma')| < 2g-2+n$

Symmetry is implied by the Lie ideal condition

3. Airy structures from W-algebras

Strategy to construct Airy structures : look at VOAs that

1. consider a VOA that admit a free field representation (i.e. in some $\mathcal{D}_{E[[\zeta]],\hbar}$)
2. identify some gr. Lie ideal of the algebra of modes
3. conjugate the representation to match the degree one condition for generators of such an ideal

This can be carried out at least for $\mathcal{W}(\mathfrak{g})$ at critical level, when $\mathfrak{g}$ = direct sum of simple, simply-laced Lie algebras

This approach finds its roots in the work of Milanov (16) and was systematised in BBCCN 18

Here we focus on $\mathfrak{g} = \mathfrak{gl}_r$

$\mathfrak{gl}_r$ Cartan algebra $\mathfrak{h} = \mathbb{C}^r$ with Killing form $\langle \cdot, \cdot \rangle$

Weyl group $\mathfrak{S}_r$

Heisenberg Lie algebra $\hat{\mathfrak{h}}_\hbar = (\mathfrak{h}[t^{\pm 1}] \oplus \mathbb{C}K) \otimes \mathbb{C}[\hbar]$
with relations $[\xi \otimes t^m, \eta \otimes t^n] = \hbar \langle \xi, \eta \rangle m \delta_{m+n,0}$ and K central

Fock space $\mathcal{F}_\hbar = \mathrm{Sym}^\bullet(\mathfrak{h}[t]) \otimes \mathbb{C}[\hbar].|0\rangle$ is a module for $\hat{\mathfrak{h}}_\hbar$
where $\xi_n := \xi \otimes t^n,\ n \geq 0$ acts by killing $|0\rangle$ and K acts by 1.

The Fock space has a structure of a VOA
$Y \ : \ \mathcal{F}_\hbar \to (\mathrm{End}\mathcal{F}_\hbar)[[t^{\pm 1}]]$
defined by $Y(\xi_{-1}, t) = \sum_{k \in \mathbb{Z}} \frac{\xi_k}{t^{k+1}}$

$$Y\big(\xi^{(1)}_{-k_1} \cdots \xi^{(l)}_{-k_l}|0\rangle, t\big) = \ : \prod_{j=1}^{l} \frac{1}{(k_j - 1)!} \frac{\mathrm{d}^{k_j - 1}}{\mathrm{d}t^{k_j - 1}} Y\big(\xi^{(j)}_{-1}|0\rangle, t\big) :$$

where the normal ordering : : in a monomial pushes
negative modes to the left, positive modes to the right

3. Airy structures from W-algebras

The $\mathcal{W}(\mathfrak{gl}_r)$-VOA at critical level has many equivalent descriptions (Fateev-Lukyanov 88, Arakawa-Molev 17)

For us, it is the sub VOA of $\mathcal{F}_\hbar$ freely and strongly generated by

$w_i := e_i(\chi_{-1}^{(1)}, \ldots, \chi_{-1}^{(r)})|0\rangle$ — e_i i-th elementary symmetric polynomial

$i \in \{1, \ldots, r\}$ — $(\chi^{(j)})_{j=1}^{r}$ orthonormal basis of $\mathfrak{h} = \mathbb{C}^r$

We decompose in modes $Y(w_i, t) = \sum_{k \in \mathbb{Z}} \frac{W_{i,k}}{t^{i+k}}$

$(W_{2,k})_{k \in \mathbb{Z}}$ form a Virasoro algebra with central charge $\mathfrak{c} = r$

More generally $[W_{i,k}, W_{j,l}]$ are nonlinear combinations of $(W_{i',k'})_{i',k'}$

(W-algebra first introduced by Zamolodchikov, 85 for $r = 3$)

Let $\mathfrak{A}$ be (a certain completion of) the associative algebra generated by the modes $(W_{i,k})_{i,k}$

The constitutive ppt of VOAs give automatically two gr. Lie ideal in $\mathfrak{A}$

The vacuum ideal $\mathfrak{A}_{(1^r)}$ generated by $\big(W_{k,i} \ : \ i \ \in \{1,\ldots,r\},\ k+i-1 \geq 0\big)$

The conformal ideal $\mathfrak{A}_{(r)}$ generated by $\big(W_{k,i} \ : \ i \ \in \{1,\ldots,r\},\ k \geq 0\big)$

We can in fact construct more for $\mathcal{W}(\mathfrak{gl}_r)$

Lemma 4 (B., Bouchard, Chidambaram, Creutzig, Noshchenko, 18)

Let $\lambda \vdash r$ be a (weakly decreasing) partition of r

Set $\lambda(i) := \min\{j \mid \lambda_1 + \cdots + \lambda_j \geq i\}$

Then $\big(W_{k,i} \ : \ i \ \in \{1,\ldots,r\},\ k + \lambda(i) > 0\big)$ generates a gr. Lie ideal $\mathfrak{A}_\lambda \subset \mathfrak{A}$

For each $\sigma \in \mathfrak{S}_r$, there is a σ-twisted representation of the VOA $\mathcal{F}_\hbar$ in $\mathcal{D}_{V,\hbar}$

Take $\sigma = (1 \cdots r_1)(r_1 + 1 \cdots r_1 + r_2) \cdots (r - r_{d-1} + 1 \cdots r)$

Use the Fourier basis of the Cartan $v^{\mu,a} = \sum_{j=1}^{r_\mu} e^{2i\pi a j / r_\mu} \chi^{(j + r_1 + \cdots + r_{\mu-1})}$

The twisted representation in question, with $V := \bigoplus_{\mu=1}^{d} \bigoplus_{k>0} \mathbb{C}.\langle x_k^\mu \rangle$ reads

$$Y^\sigma(v_a^\mu, \zeta) = \sum_{k \in a/r_\mu + \mathbb{Z}} \frac{J^\mu_{r_\mu k}}{\zeta^{k+1}} \qquad \text{with} \qquad J_k^\mu = \begin{cases} \hbar \partial_{x_k^\mu} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -k x^\mu_{-k} & \text{if } k < 0 \end{cases}$$

This restricted to an (untwisted) representation of $\mathcal{W}(\mathfrak{gl}_r)$

by differential operators on V

The mode $W_{i,k}$ is represented by a degree i differential operator

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To match the degree one condition in Airy structures, we can break homogeneity by performing a dilaton shift $J_k \to J_k - r_\mu c_\mu \delta_{k+s_\mu,0}$

The classification of W(gl_r) Airy structures amounts to :

Classify the $(r_\mu, s_\mu)_{\mu=1}^{d}$ and $\lambda \vdash r$ for which one gets in this way

$W_{i,k} = \hbar \partial_{y_{\Pi(i,k)}} + O(2)$ and $\Pi : \mathcal{I}_\lambda \to \{1, \ldots, d\} \times \mathbb{N}^*$ is a bijection

up to some linear change of variables $(y_{\mu,k})_{\mu,k} \mapsto (x_k^\mu)_{\mu,k}$

This led to the sufficient condition of Theorem 2 (Y. Schüler's master thesis)

3. Correspondence with TR

From any such Airy structure, one can obtain other (isomorphic) ones by conjugation with

$$\hat{T} = \exp\left(\sum_{\substack{\mu\in\tilde{\mathfrak{a}}\\ k>0}} \left(\hbar^{-1} F_{0,1}\left[\begin{smallmatrix}\mu\\ -k\end{smallmatrix}\right] + \hbar^{-\frac{1}{2}} F_{\frac{1}{2},1}\left[\begin{smallmatrix}\mu\\ -k\end{smallmatrix}\right]\right)\frac{J_k^{\mu}}{k}\right),$$

$$\hat{\Phi} = \exp\left(\frac{1}{2\hbar}\sum_{\substack{\mu,\nu\in\tilde{\mathfrak{a}}\\ k,l>0}} F_{0,2}\left[\begin{smallmatrix}\mu & \nu\\ -k & -l\end{smallmatrix}\right]\frac{J_k^{\mu}J_l^{\nu}}{kl}\right).$$

Theorem 5 (B., Kramer, Schüler, 20)

The $F_{g,n}$ of the corresponding partition function are the coefs of expansion of $\omega_{g,n}$ computed by TR on a suitable basis of differentials,
for a singular spectral curve (local expansion of y and $\omega_{0,2}$ specified by $F_{0,1}, F_{0,2}$ as above)

Actually more precise : TR formula iff $\forall (i,k)\in \mathcal{I}_{\mathbf{r},\mathbf{s}} \quad W_{i,k}\cdot e^{F} = 0$

If $\mathcal{I}_{\mathbf{r},\mathbf{s}} = \mathcal{I}_{\lambda}$ for some $\lambda \vdash r$, this is an Airy structure so $\omega_{g,n}$ is symmetric

Conclusion

All these Airy structures can be built from the elementary ones by

- more general dilaton shifts (expansion of y)
- direct sums (several zeros of dx)
- conjugation by exp{quadratic diff op.} (choice of $\omega_{0,2}$)

Their partition function is therefore obtain by action of operators on products of elementary partition functions (attached to each zero of dx)

$\rightsquigarrow$ Givental-like decomposition

Conclusion

For each $\mathbf{T} = (r_\mu, s_\mu, c_\mu)_{\mu=1}^{d}$ satisfying the assumptions of Theorem 2

TR for the spectral curve $\prod_{\mu=1}^{d} (x^{s_\mu - r_\mu}(y/c_\mu)^{r_\mu} - 1) = 0$ equipped with

produces a sequence of generating series

We expect them to have an interpretation in terms of intersection theory on certain moduli spaces of curves

Thank you for your attention !

based on joint works with
Andersen, Chekhov, Orantin : 1703.03307
Bouchard, Chidambaram, Creutzig, Noshchenko: 1812.08738
Kramer, Schüler : to appear