Algebra, geometry and physics seminar

Geometry of the combinatorial Teichmüller space (first part)

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based on joint work with
Andersen, Charbonnier, Giacchetto, Lewanski, Wheeler, to appear

Σ will usually denote a smooth bordered surface
oriented, connected (unless specified), genus g
n labeled boundaries $\partial_1\Sigma, \ldots, \partial_n\Sigma$
stable : $2 - 2g - n < 0$

I. Two Teichmüller spaces

II. Cutting, twisting, gluing

Interlude — Symplectic structure

III. Flowing from hyperbolic to combinatorial

IV. Changing pairs of pants

I

Hyperbolic and combinatorial Teichmüller spaces

- The **Teichmüller space** of the bordered surface Σ is defined as

$$\mathcal{T}_\Sigma = \left\{ \begin{array}{c} \text{diffeo. } f \,:\, S \to \Sigma \\ S \;\; \text{bordered Riemann surface} \end{array} \right\} \Big/ \sim$$

$(f, S) \sim (f', S')$ when

$$\begin{array}{ccc} DS & \overset{\Phi}{\dashrightarrow} & DS' \\ & f \searrow \quad \swarrow f' & \\ & D\Sigma & \end{array}$$

Φ biholomorphic, preserves ∂_i

$f' \circ \Phi \circ f^{-1}$ isotopic to id

among diffeo. preserving $\partial_i \Sigma$

- The pure mapping class group

$$\mathrm{Mod}_\Sigma^\partial = \{\varphi \in \mathrm{Diff}_\Sigma \;\; | \;\; \forall i \;\; \varphi(\partial_i \Sigma) = \partial_i \Sigma\}/\mathrm{Diff}_0(\Sigma)$$

acts on $\mathcal{T}_\Sigma$ with finite stabilizers

- The **moduli space** of bordered surfaces is defined as $\mathcal{M}_\Sigma(L) = \mathcal{T}_\Sigma(L)/\mathrm{Mod}_\Sigma^\partial$

- It can be described via hyperbolic geometry

$$\mathcal{T}_\Sigma \cong \left\{ \begin{array}{c} \text{hyperbolic metrics } \sigma \text{ on } \Sigma \\ \text{with geodesic boundaries} \end{array} \right\} \Big/ \mathrm{Diff}_0(\Sigma) \quad \circlearrowleft \ \mathrm{Mod}^\partial_\Sigma$$

Simple closed curve $\gamma \rightsquigarrow$ hyperbolic length $\ell(\gamma) : \mathcal{T}_\Sigma \to \mathbb{R}_+$

- $\mathcal{T}_\Sigma$ is a smooth manifold of dimension $(6g - 6 + 2n) + n$

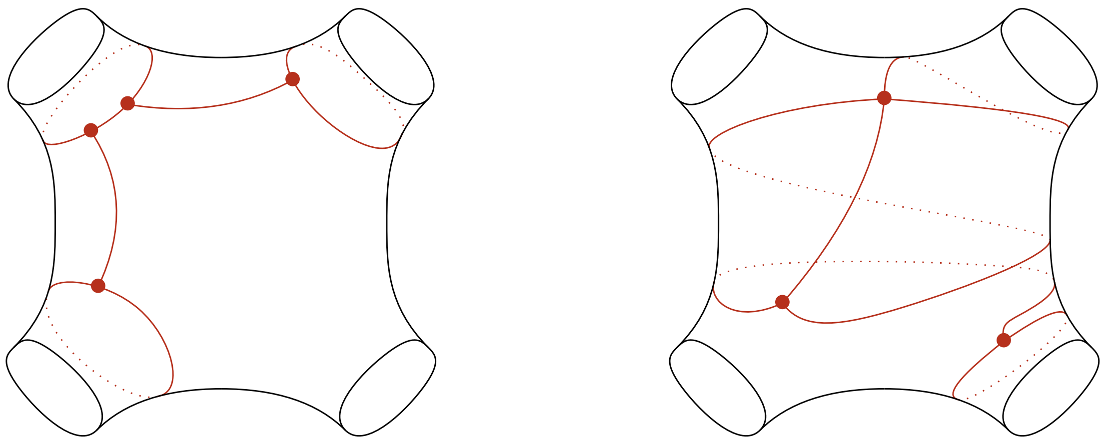
(Thurston) $\mathcal{T}_\Sigma \hookrightarrow \mathbb{R}_+^{\mathcal{S}_\Sigma}$ is an embedding $\qquad \mathcal{S}_\Sigma = \{\text{simple closed curves}\}$

($9g - 9 + 3n$ well-chosen curves suffice)

- Loci with boundary lengths $L \in \mathbb{R}_+^n$ are denoted $\mathcal{T}_\Sigma(L)$, $\mathcal{M}_\Sigma(L)$

- On moduli space we dont have length functions for non-boundary curves

A **ribbon graph** is a graph with
- the data of a cyclic order at each vertex
- vertices have valency ≥ 3
- faces are labeled from 1 to n



Combinatorial Teichmüller space

$$\mathcal{T}_\Sigma^{\mathrm{comb}} = \left\{ \begin{array}{l} \text{isotopy class of proper embeddings of metric ribbon graphs} \\ \mathbb{G} \overset{f}{\hookrightarrow} \Sigma \quad \text{such that } \Sigma \text{ retracts onto } f(\mathbb{G}) \text{ and labels agree} \end{array} \right\} \circlearrowleft \mathrm{Mod}_\Sigma^\partial$$

pure mapping class group

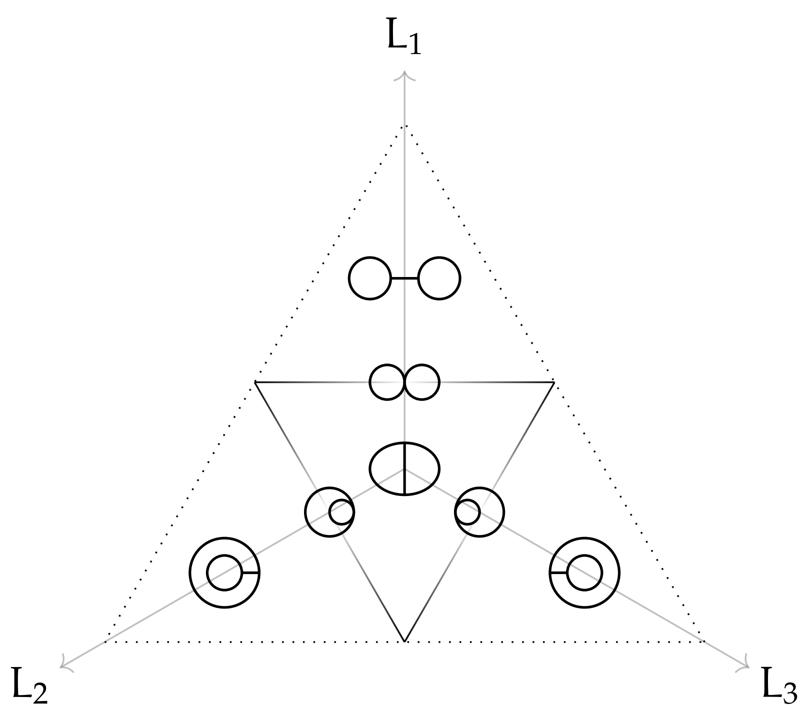
Combinatorial moduli space

$$\mathcal{M}_\Sigma^{\mathrm{comb}} = \frac{\mathcal{T}_\Sigma^{\mathrm{comb}}}{\mathrm{Mod}_\Sigma^\partial} = \bigcup_{\substack{G \text{ ribbon graph} \\ \text{type } (g,n)}} \frac{\mathbb{R}_+^{E(G)}}{\mathrm{Aut}\, G}$$

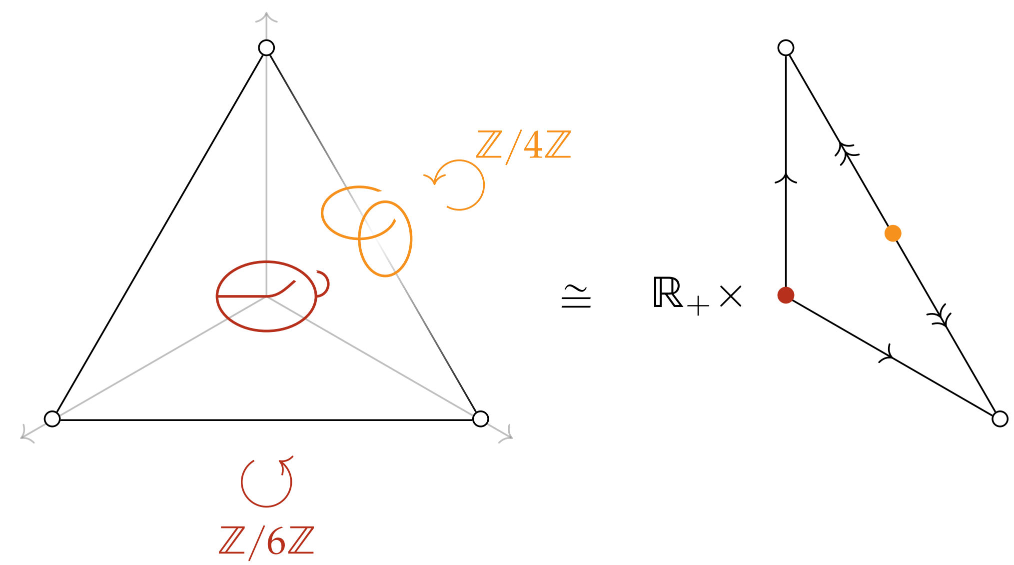
cells are glued when related by edge contraction $\rightsquigarrow$ topology

Examples of combinatorial moduli spaces

$(g,n) = (0,3)$



$(g,n) = (1,1)$



$\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$, $\mathcal{M}_\Sigma^{\mathrm{comb}}(L)$ loci with boundary lengths $L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n$

They are not smooth spaces, but rather polytopal complexes

Topology on $\mathcal{T}_\Sigma^{\mathrm{comb}}$

Consider the simplicial complex $\mathcal{A}_\Sigma$

vertices non-trivial isotopy class of proper embeddings (i.e endpoints on $\partial\Sigma$) of unoriented arcs

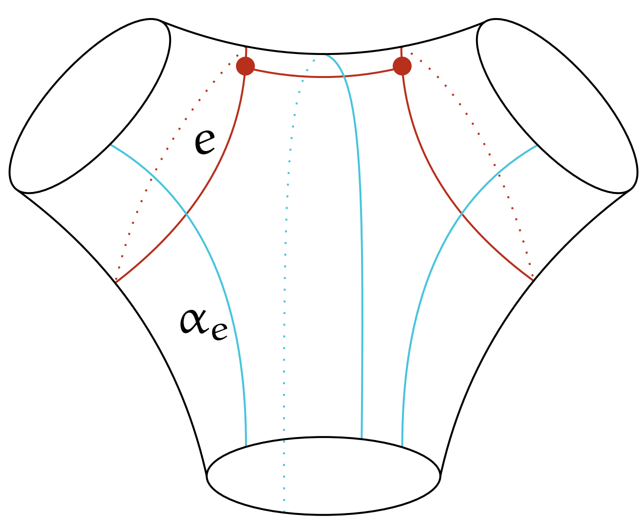
simplices classes of non-intersecting arcs $\alpha_1, \ldots, \alpha_k$ such that all components of $\Sigma \setminus \bigcup_{i=1}^k \alpha_i$ are simply connected

$$\begin{array}{ccc} \mathcal{T}_\Sigma & \longrightarrow & \mathrm{Cone}(\mathcal{A}_\Sigma) \\ \mathbb{G} & \longmapsto & \sum_{e \in E_G} \ell_{\mathbf{G}}(e)\, \alpha_e \end{array}$$

induces a topology on $\mathcal{T}_\Sigma^{\mathrm{comb}}$ (actually a homeomorphism)

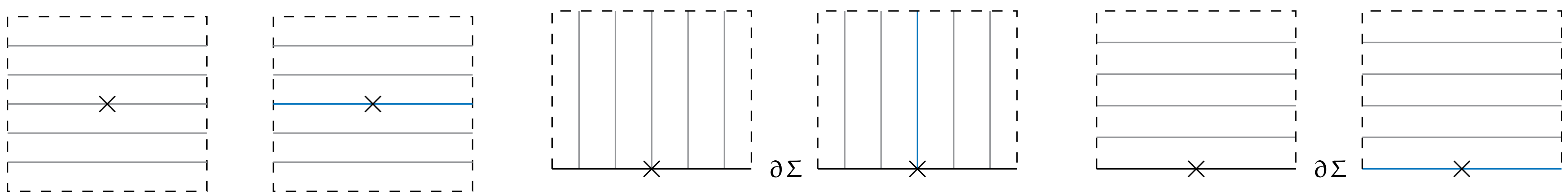
Luo 07
Mondello 09

α_e arc dual to the edge e

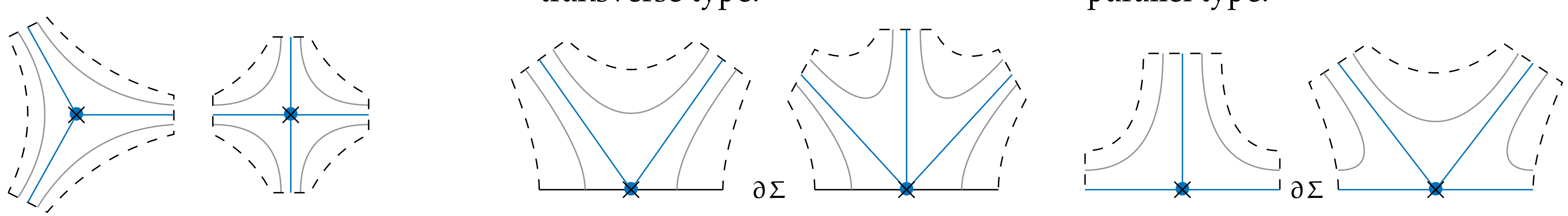


The combinatorial Teichmüller space has an equivalent description by measured foliations

$$\mathrm{MF}^{\star}_{\Sigma} = \left\{ (\mathcal{F}, \mu) \;\middle|\; \begin{array}{ll} \mathcal{F} & \text{foliation with isolated singularities} \\ \mu & \text{transverse invariant measure} \end{array} \right\} \Big/ \begin{array}{l} \text{isotopies} \\ \text{Whitehead moves} \end{array}$$

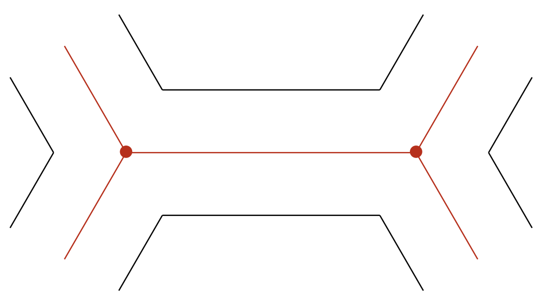


(a) Internal regular point. (b) Regular point at the boundary of transverse type. (c) Regular point at the boundary of parallel type.



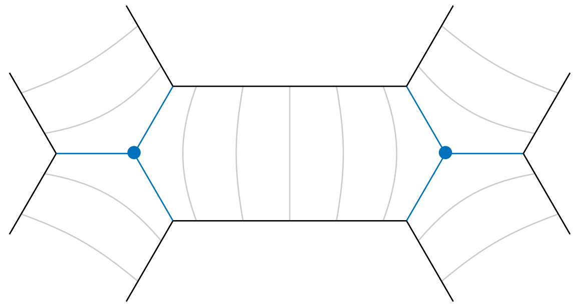
(d) Internal singular point. (e) Singular point at the boundary of transverse type. (f) Singular point at the boundary of parallel type.

The combinatorial Teichmüller space has an equivalent description by measured foliations



$$\mathcal{T}_\Sigma^{\mathrm{comb}} \hookrightarrow \mathrm{MF}_\Sigma^\star$$

homeomorphism onto its image



The image is the set of [measured foliations] where
- leaves are transverse to $\partial\Sigma$
- no saddle connections, i.e. singular leaves joining 2 singular points

Combinatorial length $\ell(\gamma) : \mathcal{T}_\Sigma^{\mathrm{comb}} \to \mathbb{R}_+$ (continuous)

- sum of edge lengths along the non-backtracking rep. on the graph

- intersection number with the measured foliation

Lemma 1 $\ell : \mathcal{T}_\Sigma^{\mathrm{comb}} \to \mathbb{R}_+^{\mathcal{S}_\Sigma}$ is an homeomorphism onto its image

Idea of proof Reconstruct arc lengths from ∂-lengths of embedded pairs of pants

Useful next week

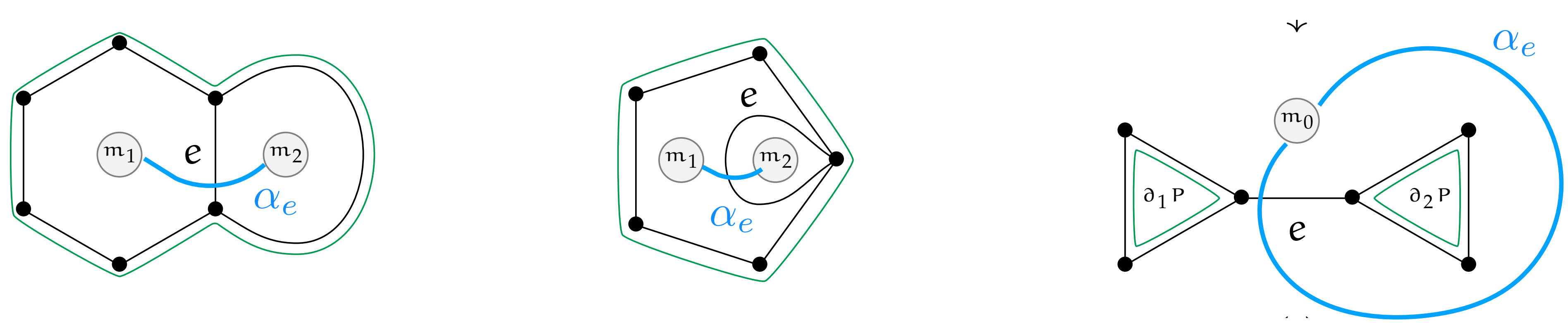
$$\mathcal{A}_\Sigma^i = \left\{ \begin{array}{c} \text{homotopy class of} \\ \alpha : [0,1] \hookrightarrow \Sigma \end{array} \;\middle|\; \alpha(0) \in \partial_i \Sigma \right\}$$

$$\mathcal{P}_\Sigma^{i,j} = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{c} \partial_1 P = \partial_i \Sigma \\ \partial_2 P = \partial_j \Sigma \end{array} \right\}$$

$$\mathcal{P}_\Sigma^i = \bigcup_{j=1}^n \mathcal{P}_\Sigma^{i,j}$$

$$\mathcal{P}_\Sigma^{i,i} = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{c} \partial_1 P = \partial_i \Sigma \\ \partial_{2,3} P \subset \mathring{\Sigma} \end{array} \right\}$$

We have a map $\mathcal{A}_\Sigma^i \longrightarrow \mathcal{P}_\Sigma^i$ and for $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$, a map $E_G^{\mathrm{or}} \longrightarrow \bigcup_{i=1}^n \mathcal{A}_\Sigma^i$



$$\mathcal{A}^i_\Sigma = \left\{ \begin{array}{c} \text{homotopy class of} \\ \alpha : [0,1] \hookrightarrow \Sigma \end{array} \;\middle|\; \alpha(0) \in \partial_i \Sigma \right\}$$

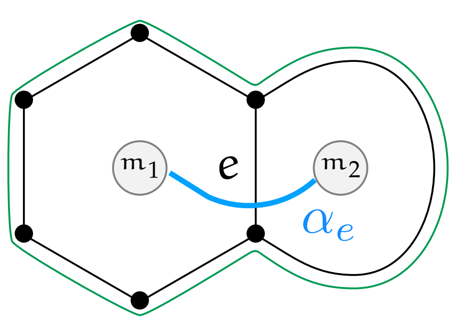
$$\mathcal{P}^{i,j}_\Sigma = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{c} \partial_1 P = \partial_i \Sigma \\ \partial_2 P = \partial_j \Sigma \end{array} \right\}$$

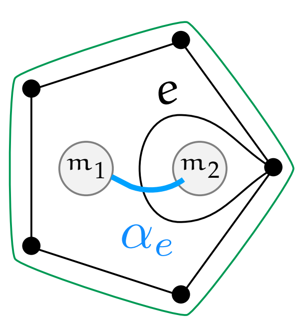
$$\mathcal{P}^i_\Sigma = \bigcup_{j=1}^n \mathcal{P}^{i,j}_\Sigma$$

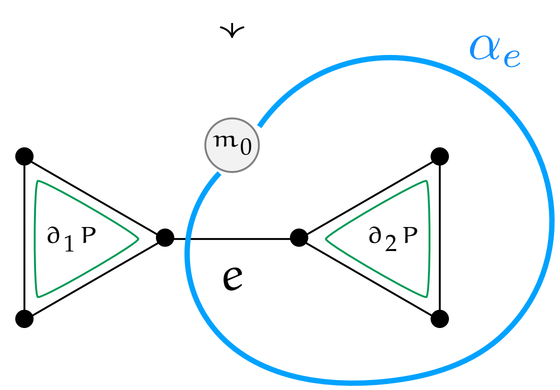
$$\mathcal{P}^{i,i}_\Sigma = \left\{ \begin{array}{l} \text{homotopy class of } P \hookrightarrow \Sigma \\ \text{such that } \Sigma - P \text{ stable} \end{array} \;\middle|\; \begin{array}{c} \partial_1 P = \partial_i \Sigma \\ \partial_{2,3} P \subset \mathring{\Sigma} \end{array} \right\}$$

Image($E^{\text{or}}_G \longrightarrow \bigcup_{i=1}^n \mathcal{A}^i_\Sigma \longrightarrow \bigcup_{i=1}^n \mathcal{P}^i_\Sigma$) = {$\mathbb{G}$-small pairs of pants}

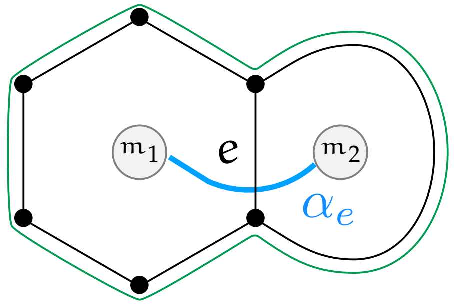
$\mathbb{G}$-small means $\ell_{\mathbb{G}}(\partial P \cap \partial\Sigma) > \ell_{\mathbb{G}}(\partial P \cap \mathring{\Sigma})$







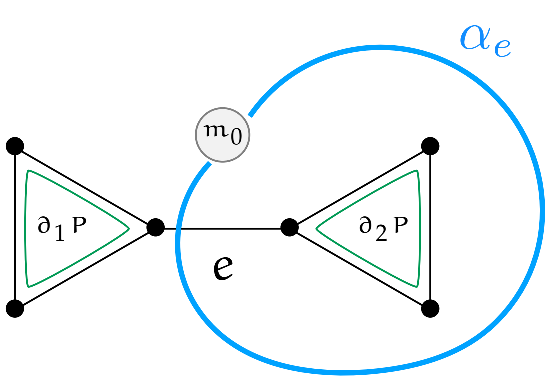
- If $[P] \in \mathcal{P}_{\Sigma}^{i,j}$ ($i \neq j$), write $\vec{\ell}_{\mathbb{G}}(\partial P) = (L_1, L_2, \ell) \leftrightarrow \mathbb{P} \in \mathcal{T}_P^{\mathrm{comb}}$



$$B_{\mathrm{K}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big([L_1 + L_2 - \ell]_+ + [L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+\big)$$

is the fraction of $\partial_1 P$ not common to $\partial_3 P$ (once retracted to $\mathbb{P}$)

> 0 iff P is $\mathbb{G}$-small



- If $[P] \in \mathcal{P}_{\Sigma}^{i,i}$, write $\vec{\ell}_{\mathbb{G}}(\partial P) = (L_1, \ell, \ell') \leftrightarrow \mathbb{P} \in \mathcal{T}_P^{\mathrm{comb}}$

$$C_{\mathrm{K}}(L_1, \ell, \ell') = \frac{1}{L_1}[L_1 - \ell - \ell']_+$$

is the fraction of $\partial_1 P$ not common to $\partial_2 P \cup \partial_3 P$ (once retracted to $\mathbb{P}$)

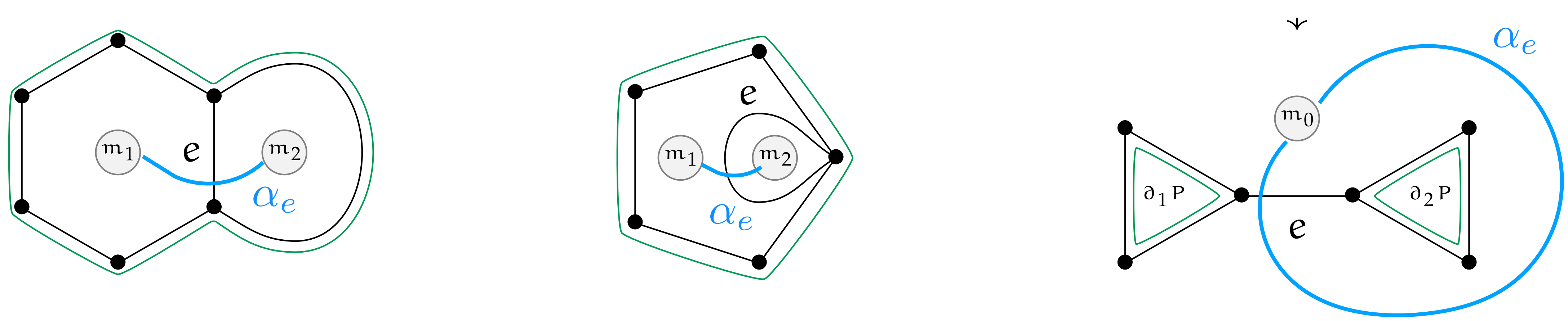
> 0 iff P is $\mathbb{G}$-small

We get a homeomorphism onto its image

$$\mathcal{T}_{\Sigma}^{\mathrm{comb}} \xrightarrow{\ell} \mathbb{R}_{+}^{\mathcal{S}_{\Sigma}} \longrightarrow (\mathbb{R}_{+}^{3})^{\bigcup_i \mathcal{P}_{\Sigma}^{i}} \longrightarrow \mathbb{R}_{\geq 0}^{\bigcup_i \mathcal{A}_{\Sigma}^{i}}$$

$$\vec{\lambda} \longmapsto \alpha \mapsto \begin{cases} \lambda(\partial_1 P_\alpha)\Big(B_{\mathrm{K}}\big(\vec{\lambda}(\partial P_\alpha)\big) - C_{\mathrm{K}}\big(\vec{\lambda}(\partial P_\alpha)\big)\Big) & \alpha \in \mathcal{A}_{\Sigma}^{i,j} \ (i \neq j) \\ \frac{1}{2}\lambda(\partial_1 P_\alpha) C_{\mathrm{K}}\big(\vec{\lambda}(P_\alpha)\big) & \alpha \in \mathcal{A}_{\Sigma}^{i,i} \end{cases}$$

In particular, we can detect which arcs are dual to edges in $\mathbb{G}$ (those having non-zero coordinates in the target)

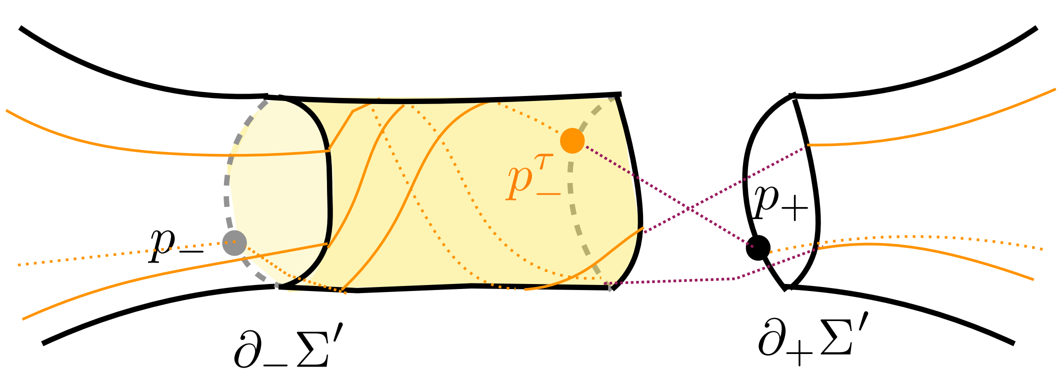


II

Cutting, twisting, gluing

- If $\gamma \subset \mathring{\Sigma}$ is a simple closed curve such that $\Sigma - \gamma$ is stable and $\sigma \in \mathcal{T}_\Sigma$, we can cut and get $\sigma_{|\Sigma-\gamma} \in \mathcal{T}_{\Sigma-\gamma}$ by taking σ-geodesic representatives

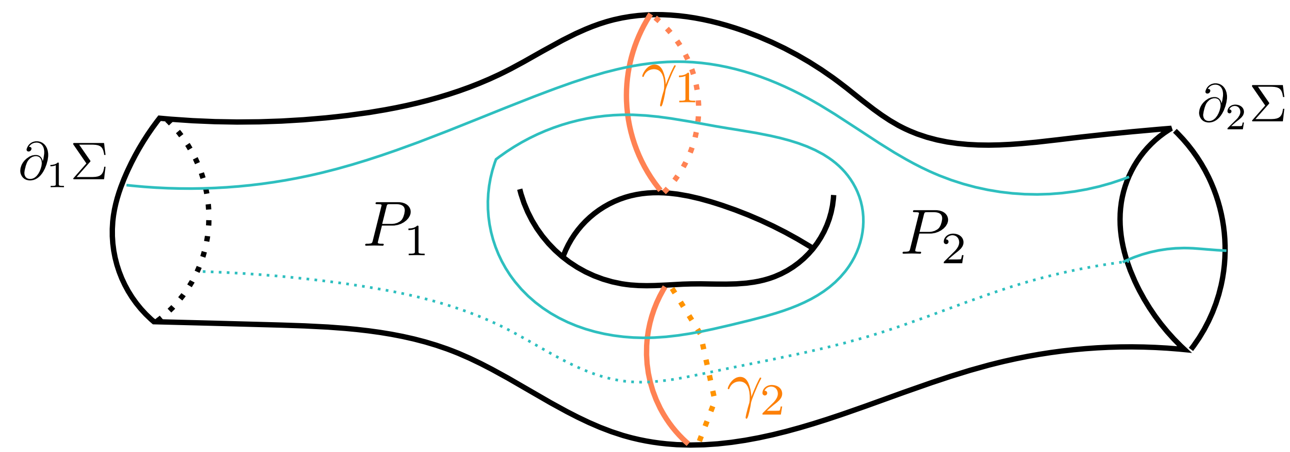
- If Σ' is a surface (possibly disconnected) with a choice of two boundaries $\partial_\pm\Sigma'$ and points $p_\pm \in \partial_\pm\Sigma'$ and $\tau \in \mathbb{R}$ and $\sigma \in \mathcal{T}_{\Sigma'}$ assigning equal length to $\partial_\pm\Sigma'$ we can twist by τ, glue $(\partial_-\Sigma', p_-^\tau)$ to $(\partial_+\Sigma', p_+^\tau)$ and uniformize the metric to get $\sigma_\tau \in \mathcal{T}_\Sigma$



Fenchel-Nielsen coordinates

Take a pair of pants decomposition $P_1, \ldots, P_{2g-2+n}$ equipped with seams

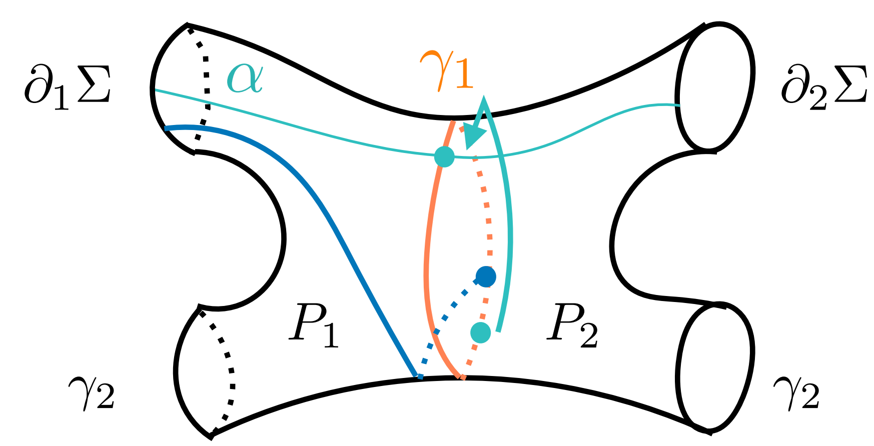
interior curves $\gamma_1, \ldots, \gamma_{3g-3+n}$



$$\begin{array}{ccc} \mathcal{T}_\Sigma(L) & \longrightarrow & (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \\ \sigma & \longmapsto & \big(\ell_\sigma(\gamma_i), \tau_\sigma(\gamma_i)\big)_{i=1}^{3g-3+n} \end{array}$$

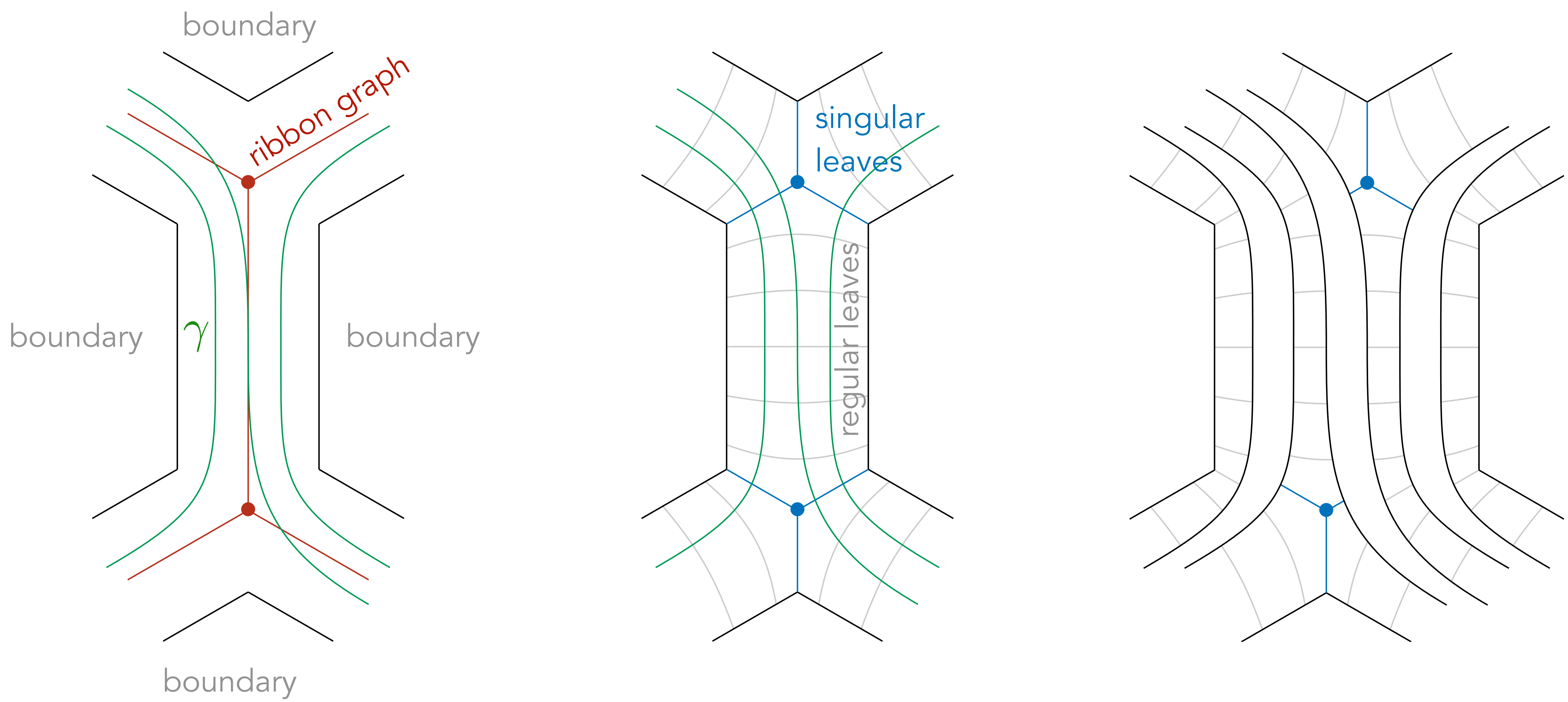
is a diffeomorphism

$\tau_\sigma(\gamma) \in \mathbb{R}$ measures the offset of a seam α crossing γ compared to the **geodesics** between the boundary it connects in the two adjacent hyperbolic pair of pants



If γ is a simple closed curve, we can cut $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}}$ along γ

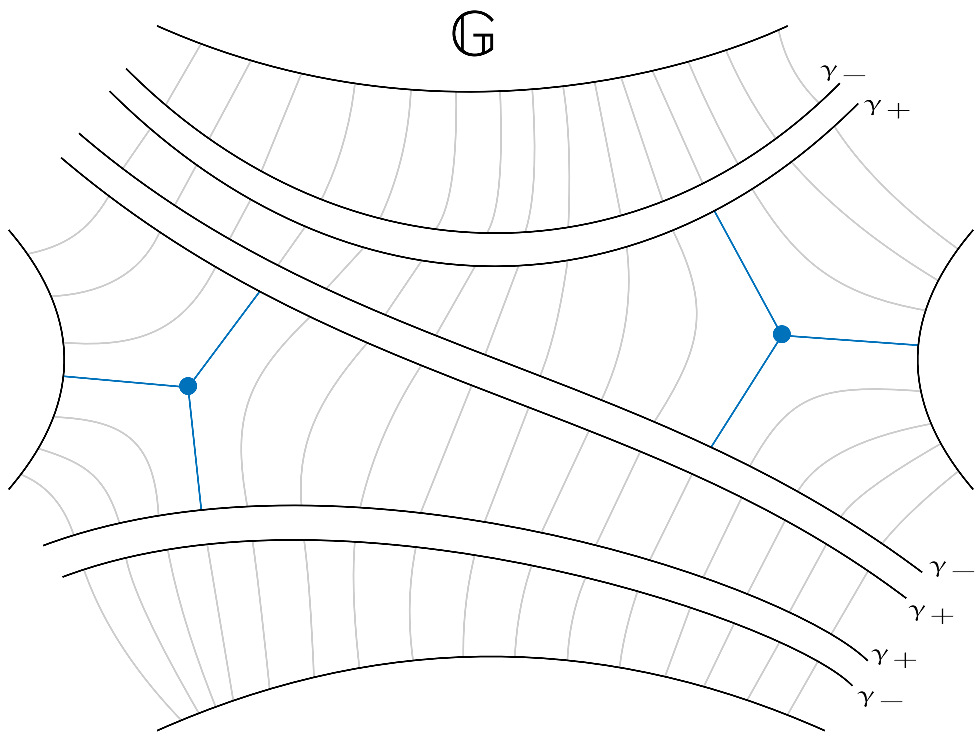
and obtain $\mathbb{G}_{|\Sigma-\gamma} \in \mathcal{T}_{\Sigma-\gamma}^{\mathrm{comb}}$

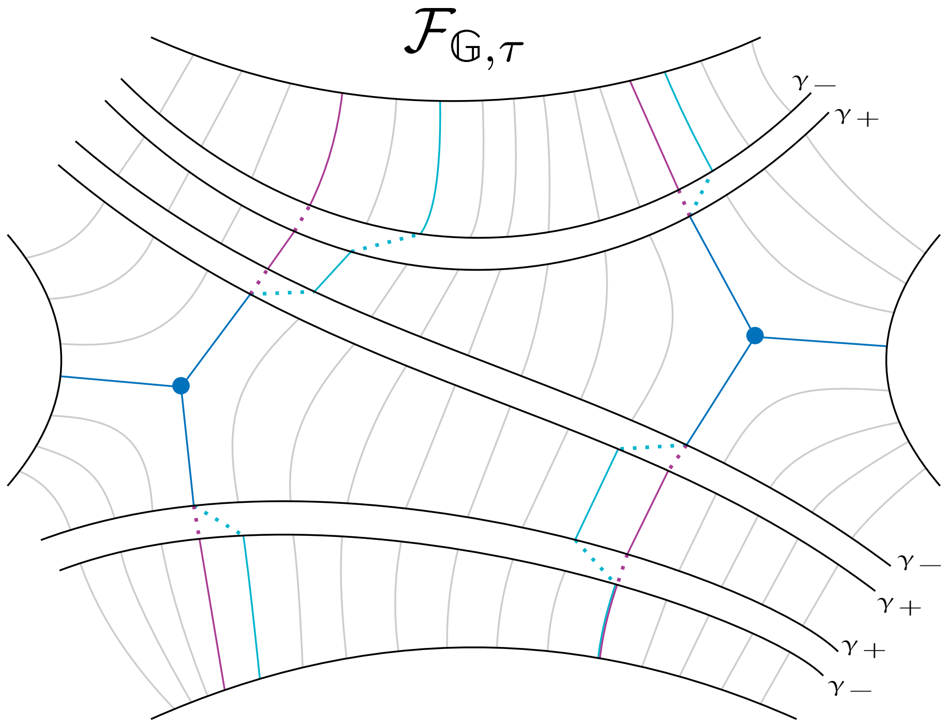


$\mathbb{G}$ $\mathbb{G}_{|\Sigma-\gamma}$

If Σ' is a surface (possibly disconnected) with a choice of two boundaries $\partial_\pm \Sigma'$ and points $p_\pm \in \partial_\pm \Sigma'$ and $\tau \in \mathbb{R}$ and $\mathbb{G} \in \mathcal{T}^{\mathrm{comb}}_{\Sigma'}$

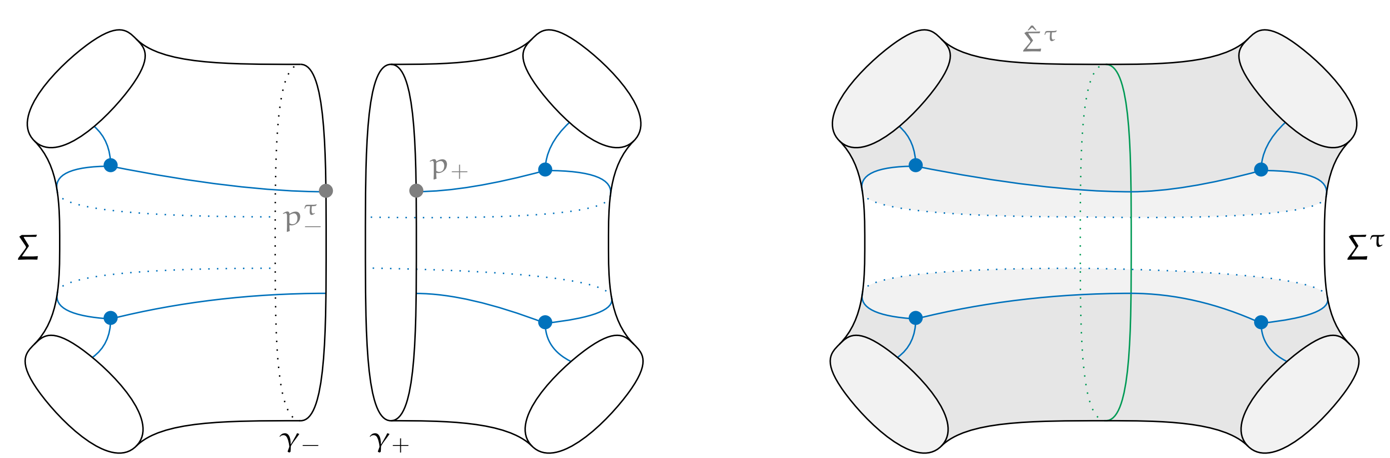
we can defined a glued surface and $\mathcal{F}_{\mathbb{G},\tau} \in \mathrm{MF}^\star_\Sigma$ by sliding p_- of the amount τ





However, $\mathcal{F}_{\mathbb{G},\tau}$ may have saddle connections

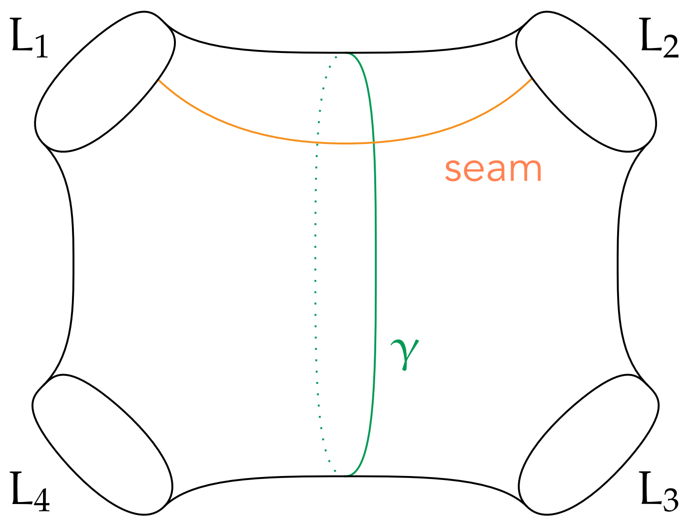
Lemma 2 $\mathcal{F}_{\mathbb{G},\tau} \in \mathcal{T}^{\mathrm{comb}}_{\Sigma}$ except for countably many τ



Take a seamed pair of pants decomposition of Σ

We have a continuous map

$$\begin{array}{ccc} \mathcal{T}_{\Sigma}^{\mathrm{comb}}(L) & \longrightarrow & (\mathbb{R}_{+} \times \mathbb{R})^{3g-3+n} \\ \mathbb{G} & \longmapsto & \big(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i)\big)_i \end{array}$$



(g,n) = (0,4) example

Theorem 3

This is an homeomorphism onto its image, which is open dense with complement of zero measure

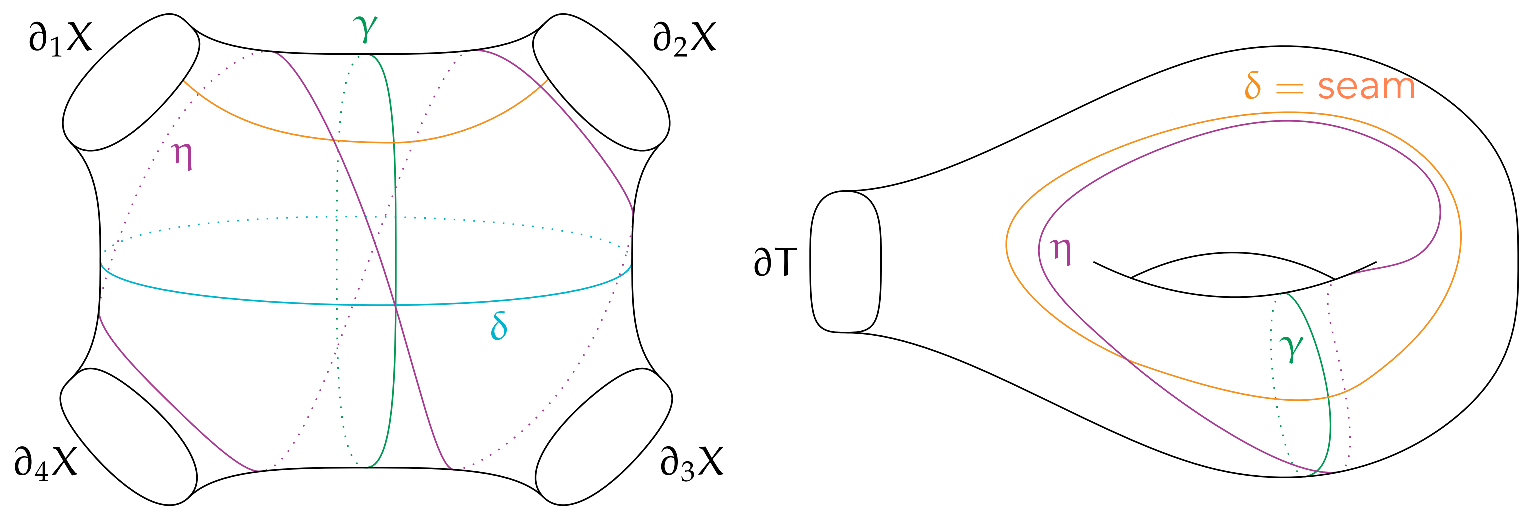
$\rightsquigarrow$ Combinatorial Fenchel-Nielsen coordinates

II.3 Cutting, twisting, gluing — Combinatorial (9g - 9 + 3n)

For each γ_i in the pair of pants decomposition, define

δ_i determined by the seam

η_i image of δ_i by a positive Dehn twist along γ_i



Combinatorial (9g - 9 + 3n)-theorem

$$\begin{array}{ccc} \mathcal{T}_\Sigma^{\mathrm{comb}}(L) & \longrightarrow & \mathbb{R}_+^{9g-9+3n} \\ \mathbb{G} & \longmapsto & \big(\ell_{\mathbb{G}}(\gamma_i), \ell_{\mathbb{G}}(\delta_i), \ell_{\mathbb{G}}(\eta_i)\big)_i \end{array}$$

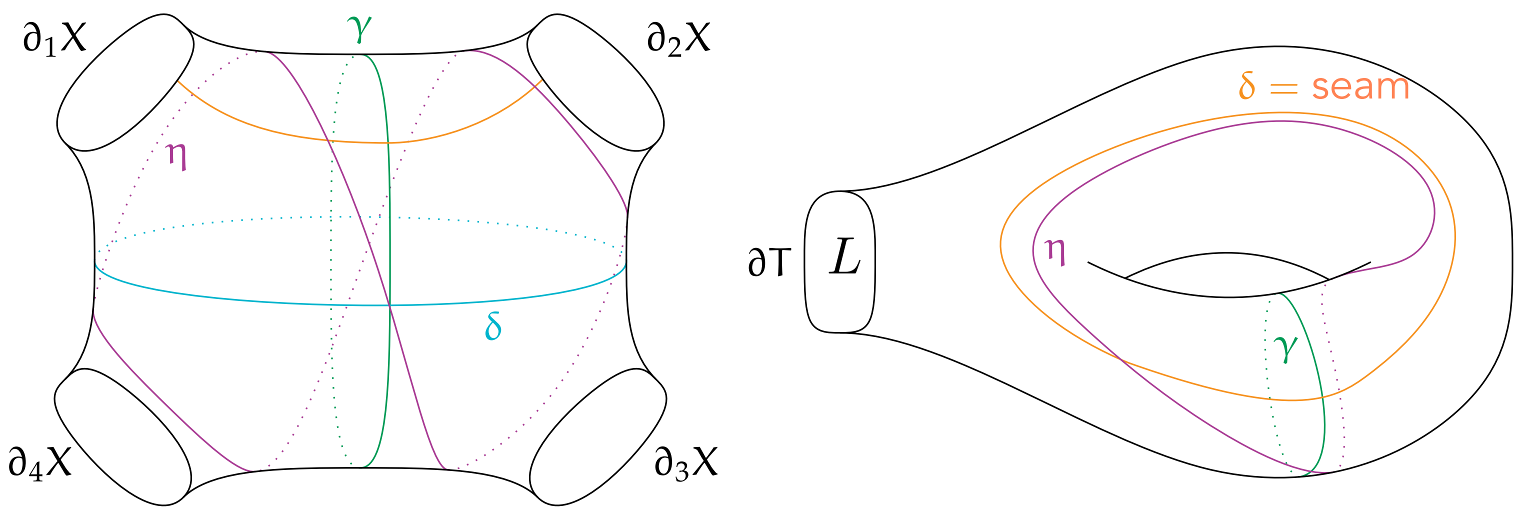
is a homeomorphism onto its image

In other words, one can express the twists in terms of lengths of simple closed curves

For each γ_i in the pair of pants decomposition, define

δ_i determined by the seam

η_i image of δ_i by a positive Dehn twist along γ_i



Idea of the proof

- In (1,1) : 4 cases (top cells for the pair of pants), where one checks

$$\begin{cases} \ell(\delta) &= |\tau(\gamma)| + \left[\frac{L}{2} - \ell(\gamma)\right]_+ \\ \ell(\eta) &= |\tau(\gamma) + \ell(\gamma)| + \left[\frac{L}{2} - \ell(\gamma)\right]_+ \end{cases}$$

inverted as $\tau(\gamma) = \frac{1}{2\ell(\gamma)}\Big(\ell(\eta) - [\frac{L}{2} - \ell(\gamma)]_+\Big)^2 - \frac{1}{2\ell}\Big(\ell(\delta) - [\frac{L}{2} - \ell(\gamma)]_+\Big)^2 - \frac{\ell(\gamma)}{2}$

- In (0,4) : 4 top cells for each pair of pants $\to$ 16 cases to discuss

Interlude —

Symplectic structure

Symplectic structure — 1. Weil-Petersson

From hyperbolic geometry, $\mathcal{T}_\Sigma(L)$ inherits a symplectic structure ω_{WP} (Weil-Petersson) which is $\mathrm{Mod}_\Sigma^\partial$ - invariant

Wolpert's formula (83) For any seamed pair of pants decomposition

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3+n} \mathrm{d}\ell(\gamma_i) \wedge \mathrm{d}\tau(\gamma_i)$$

That is : twisting along γ_i is the hamiltonian flow wrt $\ell(\gamma_i)$

Kontsevich 2-form on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$

defined on cells, $\mathrm{Mod}_{\Sigma}^{\partial}$ - invariant

$$\omega_{\mathrm{K}} = \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{e<e' \\ \text{around } \partial_i \Sigma}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{e'}$$

Lemma (Kontsevich, 91) ω_{K} is non-degenerate on cells corresponding to ribbon graphs with vertices of odd valency only

Introduced by Kontsevich in his proof of Witten's conjecture

1 - $\forall L \in \mathbb{R}_{+}^{n} \qquad \mathcal{M}_{g,n} \cong \mathcal{M}_{\Sigma}^{\mathrm{comb}}(L)$

2 - $V_{\Sigma}^{\mathrm{K}}(L) := \int_{\mathcal{M}_{\Sigma}^{\mathrm{comb}}(L)} \frac{\omega_{\mathrm{K}}^{\wedge d_{\Sigma}}}{d_{\Sigma}!} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right)$

3 - matrix model representation $\rightsquigarrow$ KdV hierarchy and Virasoro constraints

Theorem 4 For any seamed pair of pants decomposition, in each open cell

$$\omega_K = \sum_{i=1}^{3g-3+n} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i$$

$\rightsquigarrow$ combinatorial analog of Wolpert's formula (83)
for Weil-Petersson symplectic form wrt. hyperbolic length/twists

Idea of the proof

Compute the vector field ∂_{τ_i} in terms of edge lengths along γ_i (sliding)

Check it is the hamiltonian vector field for ℓ_i

Summary

$\mathcal{T}_{\Sigma}(L)$ $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$

coincide as topological spaces, but carry different geometry

$\mathcal{T}_{\Sigma}(L)$	$\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$
{hyperbolic metrics} = {marked Riemann surfaces}	subset of {measured foliations} = {marked metric ribbon graphs}
smooth manifold	polytopal complex
hyperbolic length functions (9g - 9 + 3n) embedding theorem	*combinatorial length functions* (9g - 9 + 3n) embedding theorem
hyperbolic Fenchel-Nielsen Darboux coords. for ω_{WP} full image in $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$	*combinatorial Fenchel-Nielsen* Darboux coords. for ω_{K} image = $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \setminus Z$

III

Flowing from hyperbolic to combinatorial

The spine of a hyperbolic metric σ is the locus of points in Σ equidistant from two boundaries

Lemma (Luo 07, Mondello 09)

$$\begin{array}{rccc} \mathrm{sp} : & \mathcal{T}_\Sigma & \longrightarrow & \mathcal{T}_\Sigma^{\mathrm{comb}} \\ & \sigma & \longmapsto & \mathrm{sp}_\sigma(\Sigma) \end{array}$$

is a $\mathrm{Mod}_\Sigma^\partial$-equivariant homeomorphism

The inverse is poorly understood …

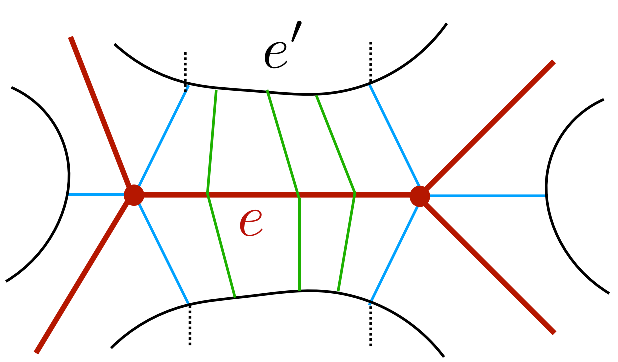
Measured foliation

— geodesics realising the equidistance

— ribs (singular leaves)

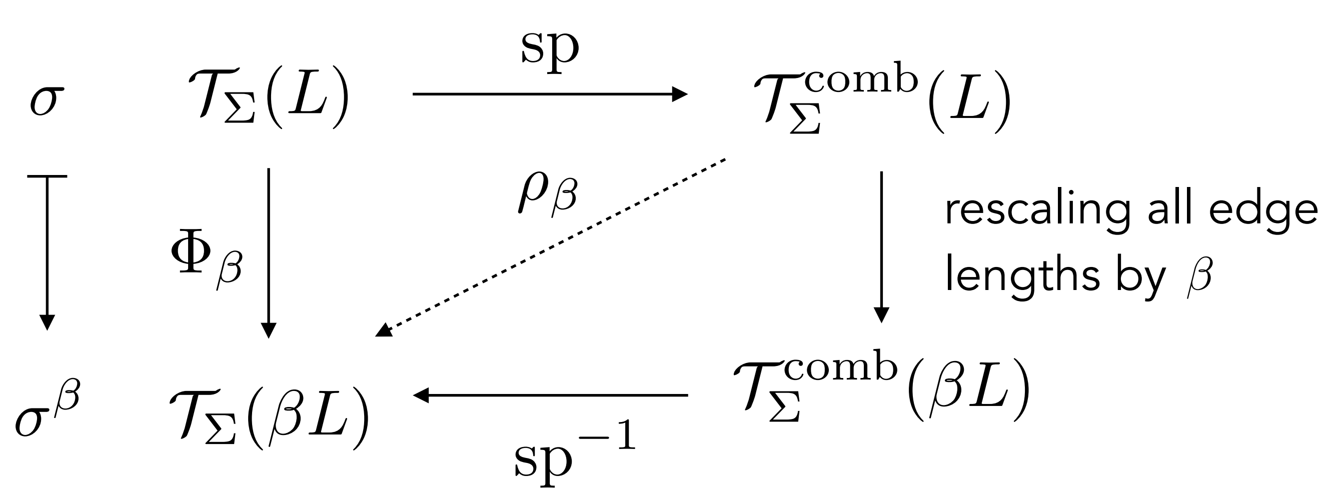
Spine

— with metric : $\ell_{\mathrm{sp}_\sigma}(e) = \ell_\sigma(e')$



Combinatorial geometry is hyperbolic geometry with large boundary lengths

Bowditch-Epstein flow (88)



The map ρ_β is not explicit …

Theorem (Mondello 09, Do 10) When $\beta \to \infty$

As metric spaces $(\Sigma, \beta^{-1}\sigma^\beta) \to \mathrm{sp}(\sigma)$ in Gromov-Hausdorff sense

$\forall \gamma \in \mathcal{S}_\Sigma$ $\beta^{-1}\ell_{\sigma^\beta}(\gamma) \to \ell_{\mathrm{sp}(\sigma)}(\gamma)$ pointwise for $\sigma \in \mathcal{T}_\Sigma(L)$

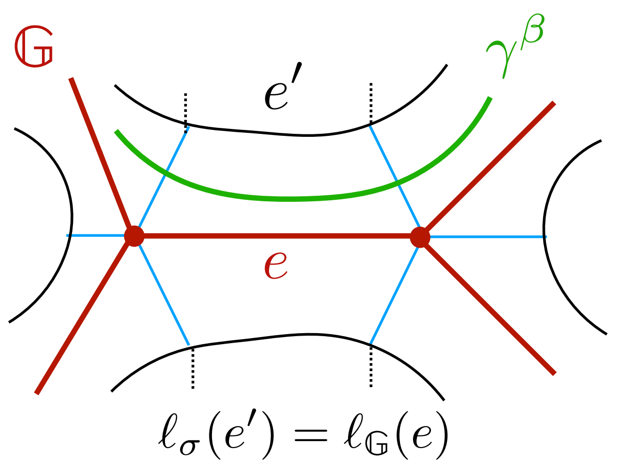
Poisson structure $\beta^2 \rho_\beta^* \Pi_{\mathrm{WP}} \to \Pi_{\mathrm{K}}$ pointwise in $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$

Lemma 5

For any $\epsilon > 0$, there is $C_{\epsilon,g,n} > 0$ such that for $\beta \geq \beta_{\epsilon,g,n}$ for any simple closed curve γ and $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$ with $\mathrm{sys}_{\mathbb{G}} \geq \epsilon$

$$\frac{\ell_{\sigma^\beta}(\gamma)}{\beta + C_{\epsilon,g,n}} \leq \ell_{\mathbb{G}}(\gamma) \leq \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \qquad \text{where } \sigma = \mathrm{sp}^{-1}(\mathbb{G})$$

Idea of the proof



- (Do, 10) Upper bound OK, and lower bound

$$\frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \leq \ell_{\mathbb{G}}(\gamma) + 2|E(\gamma)|\,\frac{r_\beta}{\beta} \qquad r_\beta = \max d_{\sigma^\beta}\big(\partial\Sigma, V(\mathrm{sp}(\sigma))\big)$$

$$E(\gamma) = \{\text{edges along } \gamma\}$$

- No cycle shorter than ϵ $\Rightarrow$ $|E(\gamma)| \leq \dfrac{c\,\ell_{\mathbb{G}}(\gamma)}{\epsilon}$
- Area bound $\mathrm{injrad}_{\sigma^\beta} = \max\big(\tfrac{1}{2}\mathrm{sys}_{\sigma^\beta}, \sup_{q\in\Sigma} d_{\sigma^\beta}(q,\partial\Sigma)\big) \leq c'$
- $\mathrm{sys}_{\sigma^\beta} \geq \beta\epsilon$ from upper bound, hence $r_\beta \leq c'$ for β large enough

Proposition 6

For each seamed pair of pants decomposition and compact $K \subset \mathcal{T}_{\Sigma}^{\mathrm{comb}}$

there exists $C'_K > 0$ such that, for $\beta \geq \beta_K$

$$\forall i \qquad \left| \frac{\tau_{\sigma^\beta}(\gamma_i)}{\beta} - \tau_{\mathbb{G}}(\gamma_i) \right| \leq \frac{C'_K}{\beta} \qquad \text{where } \sigma = \mathrm{sp}^{-1}(\mathbb{G})$$

Idea of the proof

- Use hyp. (9g - 9 + 3n)-theorem to write $\tau_{\sigma^\beta}(\gamma_i)$ in terms of hyp. lengths for σ^β
- Prove commensurable upper and lower bounds in terms of comb. lengths for $\mathbb{G}$
- Use comb. (9g - 9 + 3n)-theorem in reverse to write bounds solely with $\tau_{\mathbb{G}}(\gamma_i)$

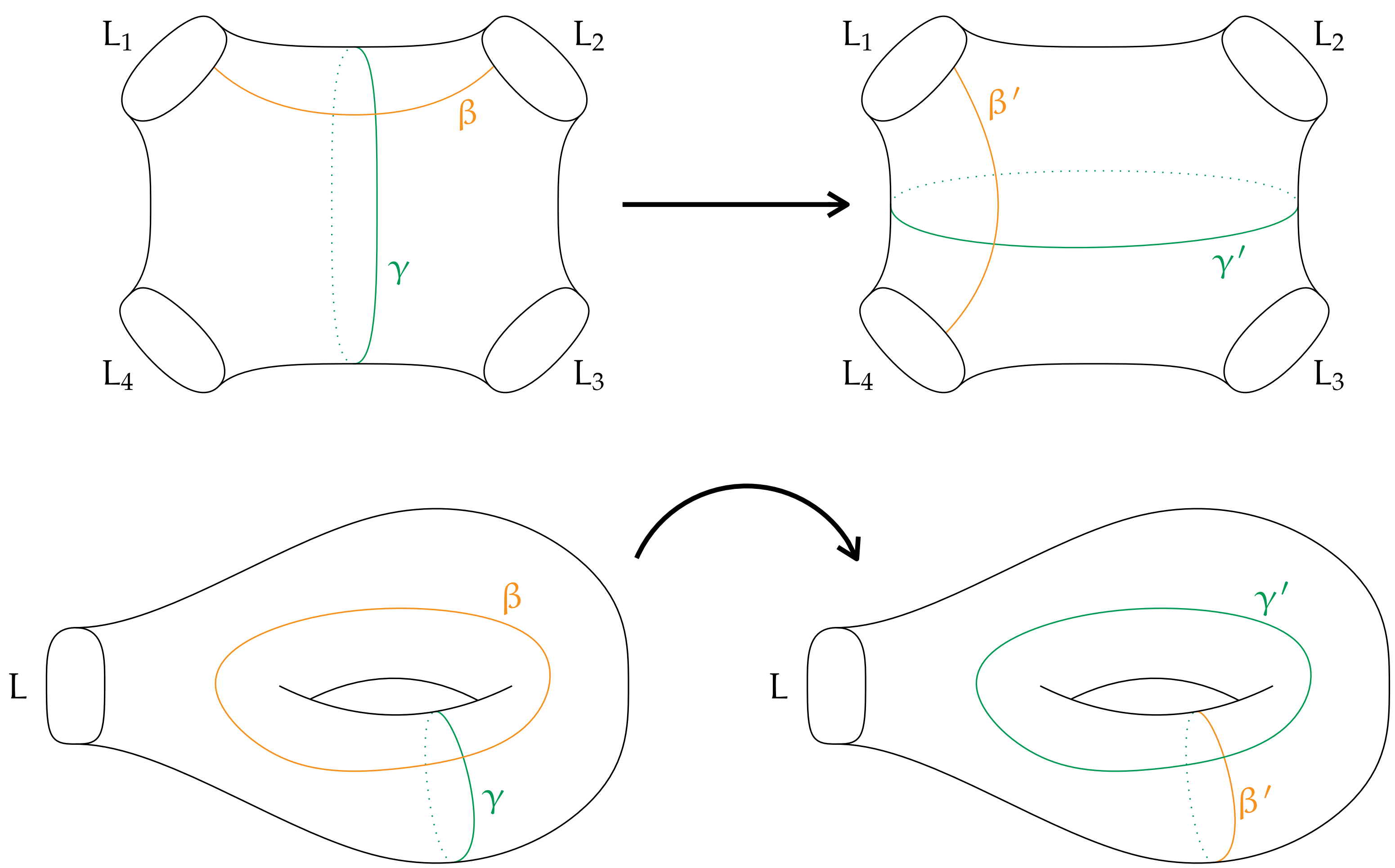
IV

Changing pairs of pants

IV Changing pairs of pants

- Fenchel-Nielsen coords. depend on a choice of a (seamed) pair of pants decompositions

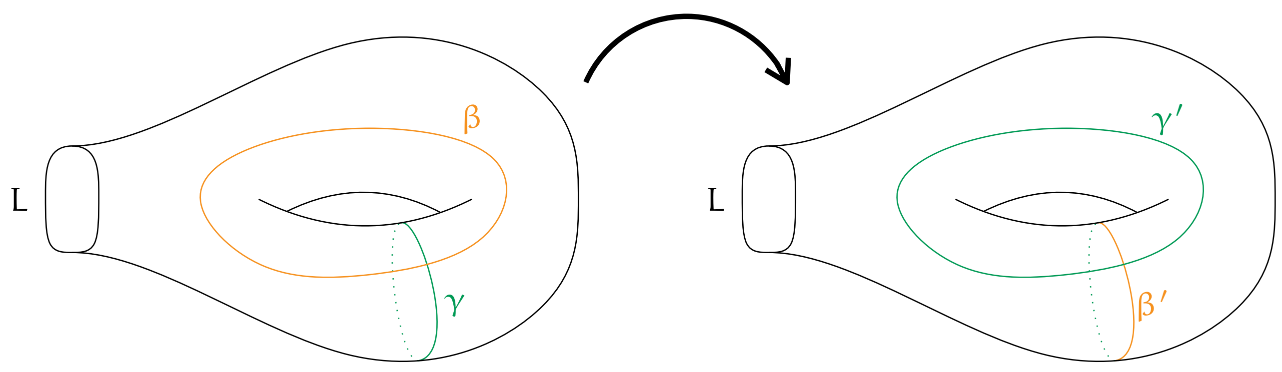
- Any two pair of pants decompositions are related by a finite sequence of flips acting on two adjacent pairs of pants or on a one-holed torus (Hatcher, Thurston)



To compute the effect of a flip, one can exploit a third description :

- $\mathcal{T}_\Sigma(L)$ is a component of $\{\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R})) \;\;|\;\; \rho(\partial_i\Sigma) \sim \mathrm{diag}(e^{L_i/2}, e^{-L_i/2})\}/\mathrm{PSL}_2(\mathbb{R})$
  using that $\mathrm{Isom}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ (and ω_{WP} is Goldman's symplectic form)

- The length of $\gamma \in \mathcal{S}_\Sigma$ is given by $2\cosh\big(\frac{\ell_\sigma(\gamma)}{2}\big) = \mathrm{Tr}\,\rho(\gamma)$



Okai's formulae (92)

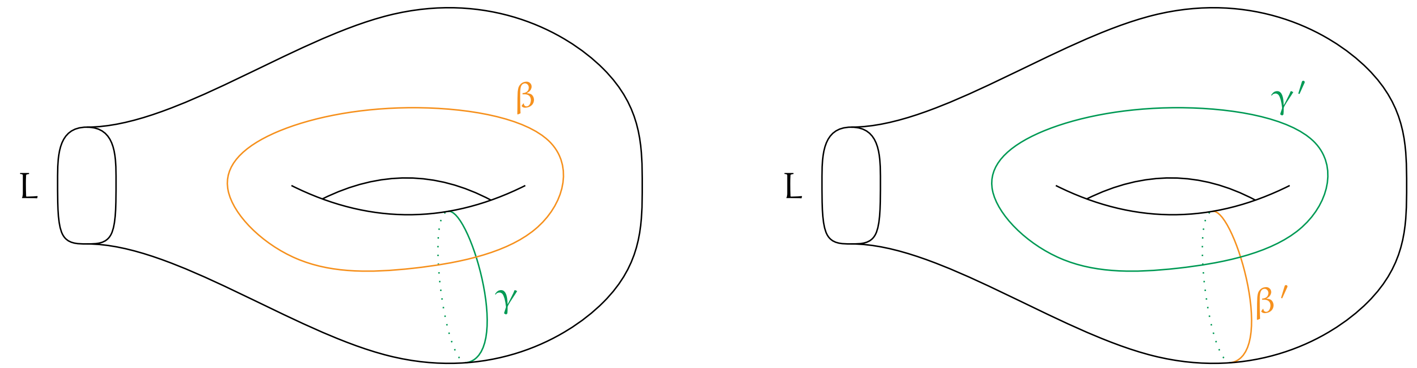
$$\begin{cases} \cosh^2\big(\frac{\ell'}{2}\big) = \dfrac{\cosh\big(\frac{\tau}{2}\big)}{\sinh\big(\frac{\ell}{2}\big)}\sqrt{\dfrac{\cosh\big(\frac{L}{2}\big) + \cosh(\ell)}{2}} \\ \cosh\big(\frac{\tau'}{2}\big) = \cosh\big(\frac{\ell}{2}\big)\sqrt{\dfrac{\cosh^2\big(\frac{\tau}{2}\big)\big(\cosh\big(\frac{L}{2}\big) + \cosh(\ell)\big) - 2\sinh^2\big(\frac{\ell}{2}\big)}{\cosh^2\big(\frac{\tau}{2}\big)\big(\cosh\big(\frac{L}{2}\big) + \cosh(\ell)\big) + \sinh^2\big(\frac{\ell}{2}\big)\big(\cosh\big(\frac{L}{2}\big) - 1\big)}} \\ \mathrm{sgn}(\tau') = -\mathrm{sgn}(\tau) \end{cases}$$

- We do not currently have a analog representation for $\mathcal{T}_{\Sigma}^{\text{comb}}(L)$ in terms of $\text{Aff}(\mathbb{R}^2) = \text{Isom}(\mathbb{R}^2)$

- It is not easy to obtain the effect of the flips on combinatorial Fenchel-Nielsen by direct computation (many cells to discuss ...)

- But we can use the flow σ^{β} $\beta \to \infty$ and the convergence results to get it

Example: flip in torus



$$\begin{cases} \cosh^2\left(\frac{\ell'}{2}\right) = \frac{\cosh\left(\frac{\tau}{2}\right)}{\sinh\left(\frac{\ell}{2}\right)}\sqrt{\frac{\cosh\left(\frac{L}{2}\right)+\cosh(\ell)}{2}} \\ \cosh\left(\frac{\tau'}{2}\right) = \cosh\left(\frac{\ell}{2}\right)\sqrt{\frac{\cosh^2\left(\frac{\tau}{2}\right)\left(\cosh\left(\frac{L}{2}\right)+\cosh(\ell)\right) - 2\sinh^2\left(\frac{\ell}{2}\right)}{\cosh^2\left(\frac{\tau}{2}\right)\left(\cosh\left(\frac{L}{2}\right)+\cosh(\ell)\right)+\sinh^2\left(\frac{\ell}{2}\right)\left(\cosh\left(\frac{L}{2}\right)-1\right)}} \\ \text{sgn}(\tau') = -\text{sgn}(\tau) \end{cases} \Longrightarrow \begin{cases} \ell' = |\tau| + \left[\frac{L}{2} - \ell\right]_+ \\ |\tau'| = -\text{sgn}(\tau)\left|\ell - \left[\frac{L}{2} - \ell'\right]_+\right| \end{cases}$$

hyperbolic   combinatorial

What remains of the smooth structure of $\mathcal{T}_\Sigma(\beta L)$ when $\beta \to \infty$:

Corollary 7

$\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$ admits a piecewise linear structure (given by comb. FN coordinates)

The transition functions are a kind of tropicalisation of Okai's formulae …

This is not implied by the “polytopal complex” structure

Summary

$\mathcal{T}_{\Sigma}(L)$ $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$

coincide as topological spaces, but carry different geometry

	flow	
{hyperbolic metrics} = {marked Riemann surfaces}	→	subset of {measured foliations} = {marked metric ribbon graphs}
smooth manifold	→	polytopal complex piecewise linear structure
hyperbolic length functions (9g - 9 + 3n) embedding theorem		*combinatorial length functions* (9g - 9 + 3n) embedding theorem
hyperbolic Fenchel-Nielsen Darboux coords. for ω_{WP} full image in $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$	→	*combinatorial Fenchel-Nielsen* Darboux coords. for ω_{K} image = $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \setminus Z$

Thank you for your attention !

Next week

Applications to

- volumes (WP or K) of moduli spaces
- volumes in the space of measured foliations

and comparison between behavior/proofs in hyperbolic and combinatorial geom.