{ geometric engineering
B-model.

$\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES

AGT
conformal field theories.

& TOPOLOGICAL RECURSION

B-model
topological strings

classical
integrability
(Painlevé equations,
quantization of Higgs
moduli space)

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joint with Vincent Bouchard
Nitin Chidambaram
Thomas Creutzig

HU QFT seminar

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① 4d Yang–Mills theory $G = SU(r)$ gauge gp.

$$\int \mathcal{D}A \, \exp\Big(-\frac{1}{2g^2}\int_{M_4} \mathrm{tr}(F_A^2)\Big)$$

A connection on G-principle bundle on M_4 $(= S_4)$.

F_A = curvature.

classical solutions : (anti) self-dual $F_A = \pm * F_A$.

modulo gauge transformation $A \to \gamma A \gamma^{-1} + \gamma^{-1} d\gamma$

$\gamma : M_4 \to G$.

instantons

instanton number $d = -\frac{1}{16\pi^2}\int_{M_4} F_A \wedge * F_A \in \mathbb{Z}$.

for (A)SD. $= c_2(\text{bundle})$.

$\rightsquigarrow \exp\left(-\frac{8\pi^2}{g^2}|d|\right)$.

$\mathcal{N} = 2$ SUSY, no matter.

$$\int \mathcal{D}A\, \mathcal{D}\psi_1 \mathcal{D}\psi_2 \mathcal{D}\phi \, \exp\left[-S_{YM}(A) + \cdots\right]$$

ψ_1, ψ_2 fermionic fields.
ϕ scalar field.

ⓐ gauge field fluctuat cancel with fermions so that

$\to$ Z as integral on the moduli space of ASD instantons.

[Donaldson 84] $\left\{\begin{array}{l}\text{moduli space of ASD} \\ \text{instantons, } \# = d \\ \text{on } S^4 \text{ modulo gauge} \\ \gamma(\infty) = \mathrm{Id}\end{array}\right\} \underset{\text{homeo}}{\simeq} \left\{\begin{array}{l}\text{algebraic} \\ G_{\mathbb{C}}\text{-bundles on} \\ \mathbb{P}^2 \text{ with trivialization} \\ \text{at } \infty,\ c_2 = d\end{array}\right\}$

$:= \mathrm{Bun}_G^d$

We want to do an equivariant computation

$(z_1, z_2) \to (\lambda^{\varepsilon_1} z_1, \mu^{\varepsilon_2} z_2)$ $\lambda, \mu \in \mathbb{C}^*$

$(Q_a)_{a=1}^{rk}$ charges $U(1)^{rk} \hookrightarrow SU(r) \subseteq G_{\mathbb{C}} = SL(r, \mathbb{C})$.

shrunk instantons to points :

Uhlenbeck space $\mathcal{U}_G^d = \bigsqcup_{d'=0}^{d} \mathrm{Bun}_G^{d} \times \mathrm{Sym}^{d-d'}(\mathbb{C}^2)$.

$(\cdots)$ $Z = \langle G | G \rangle = \sum_{d \geq 0} \Lambda^{2h^\vee d} \langle 1^d | 1^d \rangle$

$|G\rangle$ = Gaiotto vector.

$= \sum_{d\geq 0} \Lambda^{h^\vee d} |1^d\rangle$ $\quad |1^d\rangle \in IH^\circ_{GL(2)\times G}(\mathcal{U}_G^d)$

$h^\vee$ = dual coxeter #

Λ energy scale.

"fundamental class".

ⓑ SUSY localization on moduli space.

Z = sum over Young diagrams.

ⓒ 5d $\mathcal{N}=1$ gauge theory. $\mathbb{C}^2 \times [0, \beta)/\sim$

$(z_1, z_2, 0) \sim (e^{i\beta\varepsilon_1} z_1, e^{i\beta\varepsilon_2} z_2, \beta)$ Ω-background.

$\mathcal{H}_{QFT}$ Z = supersymm. index on $\mathcal{H}_{QFT}$.

4d limit $\beta \to 0 \rightsquigarrow \Lambda$ energy scale
relating $g_{YM}^{5d}, g_{YM}^{4d}, \Lambda$.

Eventually, we want to compute

$$Z = \exp\Big[\sum_{g \in \frac{1}{2}\mathbb{N}} \hbar^{g-1} \mathcal{F}_g\Big]$$

$\mathcal{F}_0$ prepotential.

"topological expansion"

$\hbar = -\varepsilon_1 \varepsilon_2$

$\alpha_0 = \varepsilon_1 + \varepsilon_2$.

$\hbar \to 0$

$\left[\begin{array}{ll}\alpha_0 \to 0 & \varepsilon_1, \varepsilon_2 \to 0 \\ \alpha_0 \text{ finite} & \text{Nekrasov–Shatashvili regime}\end{array}\right]$

② Relation to CFT ($\sim$ AGT).

$SU(r), SL(r)$
$\downarrow \quad \downarrow$
$U(r), GL(r)$.
$Q_1, \dots, Q_r$
$\left[\sum_{a=1}^{r} Q_a = 0 \text{ for } SU/SL\right]$.

- Free boson algebra for $GL(r)$.

$$J\binom{a}{z} = \sum_{m \in \mathbb{Z}} \frac{J_m^a}{z^{m+1}} \qquad a = 1, \dots, r$$

$$[J_m^a, J_n^b] = \delta_{a,b}\, m\, \delta_{n+m,0}\, \hbar$$

Representation on bosonic Fock space

$\mathcal{V} = \mathbb{C}[[t_m^a \;\; {}^{a=1,\dots,r}_{m>0}]][[\hbar^{1/2}]]$ $\quad 1 = |0\rangle$

$$J_m^a = \begin{cases} \hbar\,\partial_{t_m^a} & m > 0 \\ Q_a + \frac{\alpha_0}{2}\varepsilon_1 + \varepsilon_2 & m = 0 \leftarrow \\ -m\, t_{-m}^a & m < 0 \end{cases}$$

- $W(gl_r)$-algebra is a (chiral) CFT with primary

$$W^i(z) = \sum_{m \in \mathbb{Z}} \frac{W_m^i}{z^{m+i}}$$

$i = 1, \dots, r$
conformal weight i
(= spin).

defined by quantum Miura transform (Fateev, Lukyanov 88).

$$\sum_{i=1}^{r} W^i(z)(\alpha_0 \partial_z)^{r-i} = \prod_{a=1}^{r}{}^{\uparrow} \left(\alpha_0 \partial_z + J\binom{a}{z}\right)$$

$W^2(z)$ energy-momentum tensor

$$c = r - 1 + r(r^2-1)\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}$$

Thm (Schiffman–Vasserot 13, Braverman–Finkelberg–Nakajima 16)

① $\prod_{d \geq 0} IH^\circ_{eq}(\mathcal{U}_G^d) \simeq \mathcal{V}$ carries a representation of $W(gl_r)$

② The Gaiotto vector is uniquely determined by

$W_m^i |G\rangle = \Lambda\, \delta_{i,r}\, \delta_{m,1} |G\rangle$ [Whittaker vector]. $\quad \forall m > 0$, $i = 1, \dots, r$.

③ $\langle 0 | 0 \rangle = 1$

$(J_m^a)^\dagger = -J_{-m}^a + 2\alpha_0$ (determines the pairing).

③ Topological recursion

- $\mathcal{D}_{\mathcal{V}} = \mathbb{C}\langle\langle \underbrace{t_m^a}_{1}, \underbrace{\hbar\partial_{t_m^a}}_{1}, \underbrace{\hbar^{1/2}}_{1} \rangle\rangle$ deg = 1 $\qquad \deg \hbar = 2$, $\deg t_m^a = 1$

- An Airy structure (Kontsevich–Soibelman) is a collection $(H_m^a)_{a=1,\dots,r;\, m>0}$ of elements of $\mathcal{D}_{\mathcal{V}}^{\hbar}$

① $H_m^a = 0 + J_m^a + (\deg \geq 2)$. $\leftarrow$ easy

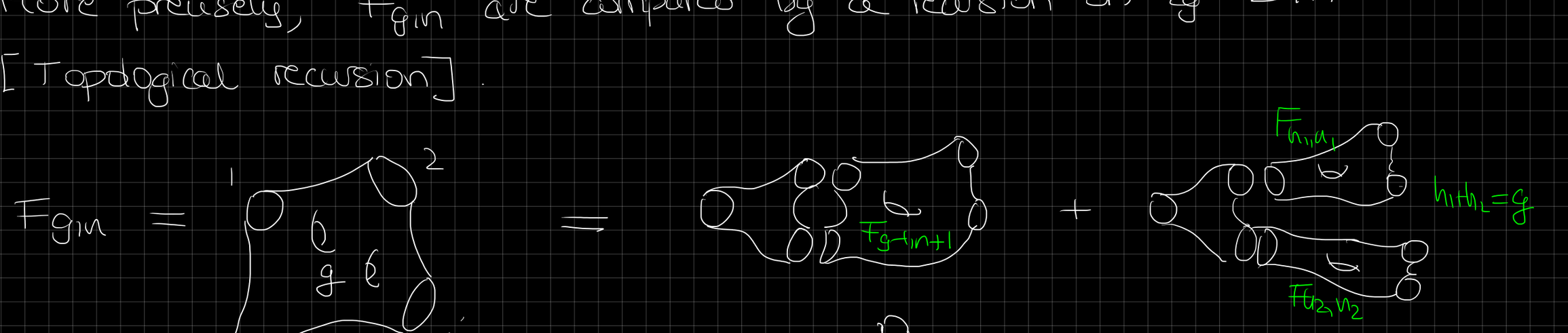
② $[H_m^a, H_n^b] \in \hbar\, \mathcal{D}_{\mathcal{V}} \cdot \mathrm{span}(H_k^c \;\; {}^{c=1,\dots,r}_{k>0})$ $\leftarrow$ rep-th

- KS thm exists a unique

$$F = \sum_{\substack{g \in \frac{1}{2}\mathbb{N} \\ n \geq 1 \\ \boxed{2g-2+n>0}}} \frac{\hbar^{g-1}}{n!} F_{g,n} \qquad F_{g,n} = \sum_{\substack{\vec{a} \in [r]^n \\ \vec{m} \in (\mathbb{N}^*)^n}} F_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix} t_{m_1}^{a_1} \cdots t_{m_n}^{a_n}$$

so that $\forall\, {}^{a=1,\dots,r}_{m>0}$ $H_m^a e^F = 0$

More precisely, $F_{g,n}$ are computed by a recursion on $2g-2+n$ [Topological recursion].



- W_m^i $\quad {}^{m>0}_{i=1,\dots,r}$ (up to linear combination) H_m^a

is an AS. $\to F = 0$

- $\Big(\underbrace{W_m^i - \overbrace{\Lambda\,\delta_{i,r}\,\delta_{m,1}}^{\deg 0}}\Big)|G\rangle = 0$

satisfies condition ② because W_1^r does not appear in commutators of positive modes [rep. th.].

does not satisfy ①.

- $W_m^i - \tilde{\Lambda}\hbar^{r/2}\delta_{i,r}\delta_{m,1}$ $\qquad \Lambda = \tilde{\Lambda}\hbar^{r/2}$ $(\to 0)$

forms an AS $\to |G\rangle$ is computed by TR.

$$Z = \langle G | G \rangle = \langle 0 | (e^F)^\dagger e^F | 0 \rangle \qquad |G\rangle = e^F \text{ contains } J_m^a\ m<0$$

$$= \langle 0 | (e^F)^\dagger e^F (e^F)^\dagger \underbrace{(e^{-F})^\dagger |0\rangle}_{|0\rangle}$$

$(e^F)^\dagger$ contains J_m^a $m>0$

$$= \langle 0 | \mathrm{Ad}_{(e^F)^\dagger}(e^F) | 0 \rangle$$

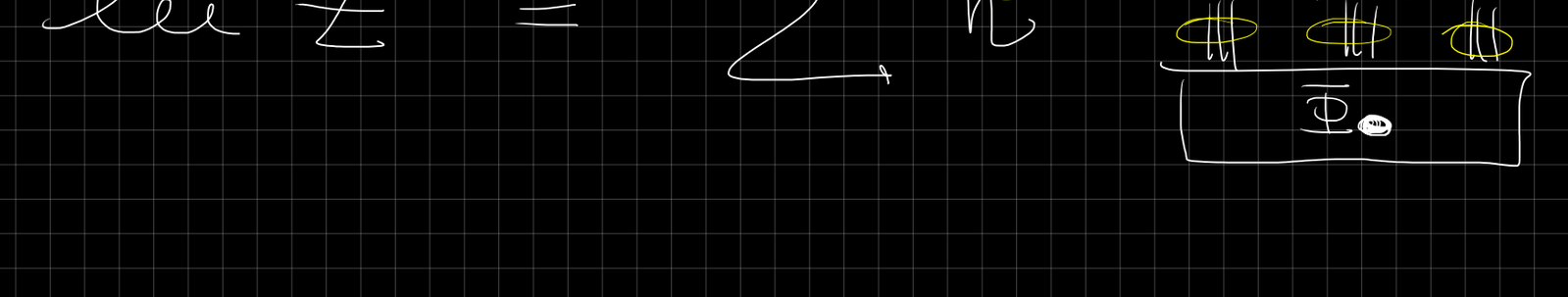
- Fact $F_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix} = \tilde{\Lambda}^{m_1+\cdots+m_r}\, \delta_{\underline{r(\sum m_i) \leq 2g}} \cdot F_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix}\Big|_{\tilde{\Lambda}=1}$

$$\hbar^{g-1} F_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix} = \hbar^{-1}\underbrace{\hbar^{g-\frac{r}{2}(\sum m_i)}}_{\hbar^{\geq 0}} \cdot *$$

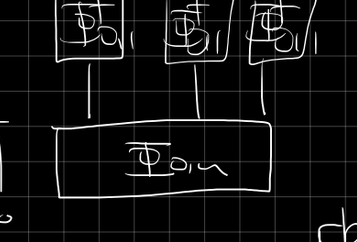
$$e^F = \exp\Big(\sum_{\substack{g \in \frac{1}{2}\mathbb{N} \\ n \geq 1 \\ \vec{a}, \vec{m}}} \hbar^{g-1}\, \Phi_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix} \Lambda^{m_1+\cdots+m_n}\Big)$$

$\Phi_{0,1} \neq 0$
$\Phi_{0,2} \neq 0$.

[ruins a priori recursion on $2g-2+n$]



each line carries some a, m



$$= \Phi_0\Big(t_k^a = \frac{\Lambda^{rk}}{k}\Phi_{0,1}\begin{bmatrix}a \\ k\end{bmatrix}\Big).$$

④ Spectral curve geometry.

$$\sum_{\vec{a}, \vec{m}} F_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix} t_{m_1}^{a_1}\cdots t_{m_n}^{a_n} \longrightarrow \omega_{g,n}\begin{pmatrix}a_1 \cdots a_n \\ z_1 \cdots z_n\end{pmatrix}$$

$$= \sum_{\vec{m}} F_{g,n}\begin{bmatrix}\vec{a} \\ \vec{m}\end{bmatrix} \prod_{i=1}^{n} \frac{dz_i}{z_i^{m_i+1}}.$$

$$= [\hbar^g] \langle 0 | J_-\binom{a_1}{z_1} \cdots J_-^{a_n}(z_n) | G \rangle_{\text{connected}}$$

$C = \bigsqcup_{a=1}^{r} \mathbb{C}_a$ spectral curve $\prod_{a=1}^{r}\left(y - \frac{Q_a}{z}\right) = 0$

$\binom{a}{z} \in C$

one reformulates TR as "B-model" computation on the spectral curve $(\varepsilon_1 + \varepsilon_2 = 0)$.

$$\omega_{g,n}\begin{pmatrix}a_1 \cdots a_n \\ z_1 \cdots z_n\end{pmatrix} = \sum_{a=1}^{r} \operatorname*{Res}_{z=0} K\begin{pmatrix}a_1 & a \\ z_1, & z\end{pmatrix}\left[\omega_{g-1,n+1}\begin{pmatrix}a & a & a_2 \cdots a_n \\ z & z & z_2 \cdots z_n\end{pmatrix} + \sum{}' \omega \cdot \omega\right]$$

when $\varepsilon_1 + \varepsilon_2 = \alpha_0 \neq 0$ (NS)

non-commutative generalisation : K is an operator expressed in terms of solutions of $\prod_{a=1}^{r}\left(\alpha_0 \partial_z - \frac{Q_a}{z}\right)\psi = 0$

$[r > 2]$

⑤ Perspectives

finite $\Lambda = \hbar^{r/2}\tilde{\Lambda}$

$\Phi_{g,n}$ $\qquad \Phi_{0,1} = y\,dz$ $\qquad C_\Lambda : \prod_{a=1}^{r}\left(y - \frac{Q_a}{z}\right) = \frac{(-1)^{r+1}\Lambda}{z^{r+1}}$

from W-constraints.

in progress : TR on C_Λ computes $\Phi_{g,n}$

see Z coupling (to all orders in g,n) by geometry of Seiberg–Witten curves. [branched cover of C_Λ]

- Further 5d ? should see emerging mirror curve of CY3 engineered from gauge theories

retrieve "remodeling the B-model" $\varepsilon_1 + \varepsilon_2 = 0$.

extend it NS. $\qquad \varepsilon_1 + \varepsilon_2 \neq 0$
(refined top. strings)