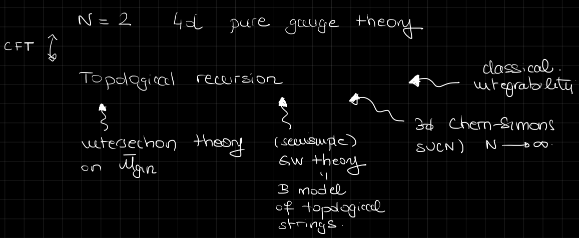
TOPOLOGICAL RECURSION IN 4D $N=2$ SUSY GAUGE THEORIES

joint work with Vincent Bouchard, Nitin Chidambaram, Thomas Creutzig (BBCC)



$N=2$ 4d pure gauge theory
CFT
Topological recursion
classical. integrability
intersection theory on $\overline{\mathcal{M}}_{g,n}$
(semisimple) GW theory
"
B model of topological strings.
3d Chern-Simons SU(N) $N \longrightarrow \infty$.

① Gauge theory

- 4d Yang-Mills $M_4 = S^4$ $G = SU_r$

$$\int \mathcal{D}A \exp\left[-\frac{1}{2g^2}\int_{M_4} \mathrm{tr}(F_A^2)\right]$$

(anti) self-dual connections $* F_A = \pm F_A$, modulo gauge transformations.

instantons

instanton number $d = -\frac{1}{16\pi^2}\int_{M_4} \mathrm{tr}(F_A \wedge * F_A) \in \mathbb{Z}.$

Donaldson 84. $\mathrm{Bun}_G^d = \left\{ \begin{array}{l} \text{algebraic } G\text{-bundles} \\ \text{with trivialization at } \infty \\ \text{on } \mathbb{P}^2, \text{ fixed } d \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{framed} \\ \text{instanton} \\ d \text{ fixed on } S^4 \end{array} \right\} / \text{gauge}.$

smooth, $\dim_{\mathbb{C}} = d \cdot r$, non compact.

Uhlenbeck space. $U_G^d \underset{\text{set}}{=} \bigsqcup_{d'=0}^{d} \mathrm{Bun}_G^{d'} \times \underbrace{\mathrm{Sym}^{d-d'}(\mathbb{A}^2)}_{\uparrow}.$ $G = SL_r.$

partial compactification stratified.

- bundles $\rightsquigarrow$ sheaf (loc. free) of sections.

weaken condition [Losev, Nekrasov, Shatashvili, 99].

torsion free sheaves. $\mathcal{F}$

$\dim_{\mathbb{C}} = 2$: $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathrm{Sky}_Z \longrightarrow 0$

$\dim_Z = 0.$

$d = c_2$

$\tilde{U}_G^d = \left\{ \begin{array}{l} \omega\text{-semistable torsion} \\ \text{free sheaves} \end{array} \right\}$ is compact.

- $N=2$ supersymmetry, contributions of fluctuat around (A)SD. can be reduced to $\tilde{U}_G^d$. (e.g localization).

$$\boxed{\begin{array}{lcl} Z_{inst.}(\mathbb{P}^2) & = & \langle G | G \rangle . \\ |G\rangle & = & \sum\limits_{d\geq 0} \Lambda^{rd}\, \underline{|1^d\rangle} \end{array}} \in \prod_{d\geq 0} \boxed{IC^{\bullet}_{\mathbb{G}}(\mathcal{U}^d_G)} \otimes_{A_{\mathbb{G}}.} \mathbb{F}_{\mathbb{G}} = M.$$

$\mathcal{U}^d_G \circlearrowleft \mathbb{G} = \underline{G} \times GL_2(\mathbb{C})$ $\quad (\lambda_1, \lambda_2) \in (\mathbb{C}^\times)^2 \quad (z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2).$

$H^{\bullet}_{\mathbb{G}}(\{pt\}) = \mathbb{C}[\underset{(\varphi_1, \dots, \varphi_r, \varepsilon_1, \varepsilon_2)}{\mathfrak{h} \times \mathbb{C}^2}]^{Weyl.} = A_{\mathbb{G}} \qquad \mathbb{F}_{\mathbb{G}} = \mathrm{Frac}(A_{\mathbb{G}})$

$|1^d\rangle$ fundamental class of $\mathcal{U}^d_G$

② AGT framework

Thm (Schiffman, Vasserot 13, SL_r, GL_r; Braverman – Finkelberg – Nakajima 16, $G =$ complex simply laced gp.)

① M has the structure of a Verma module for $\mathcal{W}^k(\mathfrak{g})$ - algebra $\quad k = \kappa + h^\vee \quad \kappa = -\varepsilon_1/\varepsilon_2$

highest-weight vector $|0\rangle$, highest weight is $(\varphi_a - (\varepsilon_1 + \varepsilon_2))_{a=1}^r$

② There exists $|1^d\rangle$ $d \geq 0$ fundamental class uniquely determined by

$W^i_m |1^d\rangle = \underline{\delta_{i,r}} \delta_{m,1} |1^{d-1}\rangle \quad \forall i \in [r] = \{1, \dots, r\}, \; m > 0.$

(Gaiotto vectors) $\left[|G\rangle = \sum_{d\geq 0} \Lambda^{h^\vee d} |1^d\rangle \right.$ is Whittaker vector $\quad \left. W^i_m |G\rangle = \underline{\delta_{i,r}\delta_{m,1}} \Lambda^{h^\vee d} |G\rangle . \right]$

③ The intersection pairing is characterized by

$\langle 0 | 0 \rangle = 1$

$(J^a_m)^\dagger = -J^a_{-m} + 2(\varepsilon_1 + \varepsilon_2)\delta_{m,0}$

$$\hbar = -\varepsilon_1 \varepsilon_2 , \quad \kappa = -\varepsilon_2/\varepsilon_1 .$$

$$\alpha = \varepsilon_1 + \varepsilon_2 = \hbar^{1/2}\underbrace{(\kappa^{-1/2} - \kappa^{1/2})} = \hbar^{1/2}\alpha_0$$

Heisenberg algebra for $\mathfrak{gl}_r$ $\quad \mathfrak{h} = \mathbb{C}^r$, o.n basis $(\alpha^a)_{a=1}^r$

generators $J^a_m \quad a = 1, \dots, r, \; m \in \mathbb{Z}$

relations $[J^a_m, J^b_n] = \delta_{a,b}\, m\, \delta_{m+n,0}\, \hbar$

$\rightsquigarrow J^a(z) = \sum\limits_{m\in\mathbb{Z}} \dfrac{J^a_m}{z^{m+1}} \rightsquigarrow$ VOA

$W^k(gl_r)$ – algebra $\quad W^i(\xi) = \sum_{m\in\mathbb{Z}} \frac{W^i_m}{\xi^{m+i}}$ $\quad i = 1,\dots,r$
[Zamolodchikov] $\quad$ fields of conformal weight i
83.

quantum Miura transform :

$$\sum_{i=1}^r \underbrace{W^i(\xi)}\,(\alpha\partial_\xi)^{r-i} = \overrightarrow{\prod_{a\in[r]}}\,(J^a(\xi) + \alpha\partial_\xi).$$

$$= (J^1(\xi) + \alpha\partial_\xi)(J^2(\xi) + \alpha\partial_\xi)\cdots(J^r(\xi)+\alpha\partial_\xi)$$

$$W^1(\xi) = \sum_{a=1}^r J^a(\xi).$$

$$W^2(\xi) = \sum_{m\in\mathbb{Z}} \frac{W^2_m}{\xi^{m+2}}$$

Virasoro algebra

$$c = r-1 + r(r^2-1)\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}.$$

$\vdots$

highest weight $\quad J^a_0|0\rangle = (Q_a - (\varepsilon_1+\varepsilon_2))|0\rangle.$

③ Airy structures and topological recursion

- V vector space / $\mathbb{C}$ $\quad (x_i)_{i\in I}$ linear coord on V.

$$\mathcal{D}^\hbar_V = \mathbb{C}\langle (x_i, \hbar\partial_{x_i})_{i\in I}\rangle[[\hbar^{1/2}]] \qquad [\hbar\partial_{x_i}, x_j] = \hbar\delta_{ij}.$$

graded algebra $\quad \deg x_i = 1$
$\deg \hbar = 2$
$\deg \underline{\hbar\partial_{x_i}} = 1$

Def • An Airy structure in normal form on V
is a family $H_i \in \mathcal{D}^\hbar_V$ $\; i\in I$ such that
① low degree condition $\quad H_i = \underline{\hbar\partial_{x_i}} + O(2)$
② graded subalgebra condition $\quad [H_i, H_j] \in \hbar\,\mathcal{D}^\hbar_V\,\mathrm{span}(H_k\ \ k\in I).$
$\forall i,j$

• An Airy structure (AS) is $\quad \hat{H}_i = \sum_{j\in I} N_{ij} H_j \quad N_{ij}\in\mathbb{C}.$
[basis free notion]. $\quad H_j$ is an Airy str in normal form

Thm (Kontsevich, Soibelman 17).

Given an AS $(\hat{H}_i)_{i\in I}$
there exists a unique wave function. $\quad Z = e^F$

$$F \in \hbar^{-1}\,\mathrm{Sym}^\circ V^*[[\hbar^{1/2}]]$$

$$= \sum_{\substack{g\in\frac{1}{2}\mathbb{N}\\ n>0\\ i_1,\dots,i_n\in I\\ 2g-2+n>0}} F_{g,n}[i_1,\dots,i_n]\,\frac{\hbar^{g-1}}{n!}\,x_{i_1}\cdots x_{i_n}.$$

such that $\quad \forall i\in I \quad \hat{H}_i\cdot e^F = 0$

- The $F_{g,n}$ are computed by induction on $2g-2+n > 0$.

Ex deg 2 operators

$$H_i = \boxed{\hbar \partial_{x_i}} - \sum_{a,b \in I} \boxed{\tfrac{1}{2} A^i_{a,b} x_a x_b + B^i_{a,b} \boxed{x_a} \hbar \partial_{x_b} + \tfrac{1}{2} C^i_{a,b} \hbar^2 \partial_{x_a} \partial_{x_b}} - \hbar D^i$$

$$H_i e^F = 0$$

$+$

$$F_{0,1} = 0$$
$$F_{0,2} = 0.$$

$\rightsquigarrow$ $F_{0,3}[i_1, i_2, i_3] = A^{i_1}_{i_2, i_3}$

$F_{1,1}[i_1] = D^{i_1}$

and for $2g-2+n \geq 2$.

$$F_{g,n}[\boxed{i_1}\, i_2 \cdots i_n] = \sum_{\substack{a \in I \\ m=2,\dots,n}} B^{i_1}_{i_m, a} F_{g,n-1}[a\, i_2 \cdots \hat{i}_m \cdots i_n].$$

$$+ \frac{1}{2} \sum_{a,b \in I} C^{i_1}_{ab} \Big(F_{g-1,n+1}(a b i_2 \cdots i_n) + \sum_{\substack{J \cup J' = \{i_2 \cdots i_n\} \\ h+h' = g}} F_{h,1+|J|}(a, J) F_{h',1+|J'|}(b, J') \Big).$$

"Topological recursion"

$\boxed{\text{RHS only contains } F_{g',n'} \quad 2g'-2+n' < 2g-2+n.}$

g = "genus", n = "# boundaries" in a surface.

terms $\leftrightarrow$ excision of a pair of pants in $\Sigma_{g,n}$.

generalize to higher degree

- (Milanov 15) BBCC + Noshchenko (18). One can construct AS from $\mathcal{W}(gl_r)$. [VOA having a free field representat.]

$$V = \prod_{\substack{a=1,\dots,r \\ m>0}} \mathbb{C} \cdot \langle x^a_m \rangle \qquad I = [r] \times \mathbb{Z}_{>0}.$$

$$J^a_m = \begin{cases} \hbar \partial_{x^a_m} & m > 0. \\ Q_a - \hbar^{1/2} \alpha_0 & m = 0 \\ -m x^a_{-m} & m < 0. \end{cases}$$

① low degree

$\rightsquigarrow$ $[W^i_m, W^j_n] \subseteq \hbar \mathcal{D}^{\hbar}_V \cdot \widehat{\operatorname{span}}_{\mathbb{C}} (W^k_\ell)_{(k,\ell) \in I'}$

$\forall (i,m), (j,n) \in I$

② subalgebra.

$I' = I - \{(i=r, m=1)\}.$

- wave f^n $\quad W^i_m e^F = 0 \quad \forall (i,m) \in I \quad \rightsquigarrow \quad e^F = 1 = |0\rangle$
- $\boxed{\overline{W}^i_m} = \underbrace{W^i_m} - \underbrace{\delta_{i,r} \delta_{m,1} \cancel{\Lambda}^{rd}}_{\Lambda = \hbar^{T/2} \mathbf{T}}$ still satisfy the subalgebra condition. is an AS.

$$\forall (i,m) \in I \quad \overline{W}^i_m e^{\overline{F}} = 0 \iff W^i_m e^{\overline{F}} = \delta_{i,r} \delta_{m,1} \Lambda^{rd} e^{\overline{F}}$$

Thm (BBCN 21) $\quad e^{\overline{F}} = |\mathcal{G}\rangle$ is computed by topological recursion.

- $Z_{inst}(\mathbb{P}^2) = \langle G | G \rangle.$

Lemma (proof by induction) $F_{g,n}\begin{bmatrix} a_1 \cdots a_n \\ m_1 \cdots m_n \end{bmatrix} = T^{\sum m_i}\, \tilde{F}_{g,n}\begin{bmatrix} \vec{a} \\ \vec{m} \end{bmatrix} \delta_{\underline{r(\sum m_i) \le 2g}}.$

$$F = \sum_{\substack{g,n \\ m_1,\dots,m_n \\ a_1,\dots,a_n}} \frac{\hbar^{g-1}}{n!} \underset{(\Lambda \hbar^{-r/2})^{\sum m_i}}{\boxed{T}^{\sum m_i}} \tilde{F}_{g,n} = \sum_{\substack{g \ge 0 \\ n > 0}} \frac{\hbar^{g-1}}{n!} \Phi_{g,n}(\Lambda) = \Phi$$

τ no zero $\Phi_{0,1}$ $\Phi_{0,2}.$

$\tilde{F}_{0,1} = 0$

$\tilde{F}_{0,2} = 0.$

$\underline{F_{g+(\sum m_i)\frac{2}{r}}}$

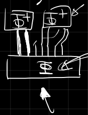
$\Phi_{0,1}$

$\Phi_{0,2}$

$$|G\rangle = e^{\Phi}|0\rangle$$

$$\begin{aligned} \langle G | G \rangle &= \langle 0 | e^{\Phi^\dagger} e^{\Phi} | 0 \rangle. \\ &= \langle 0 | e^{\Phi^\dagger} e^{\Phi} e^{-\Phi^\dagger} | 0 \rangle. \\ &= \langle 0 | \exp(\exp(\mathrm{ad}_{\Phi^\dagger}) \circ \Phi) | 0 \rangle. \end{aligned}$$

prepotential.



$$= \exp\left[\sum_{\text{graphs}} \hbar^{\text{genus}-1} \quad g \right] = \exp\left(\frac{F_0}{\varepsilon_1 \varepsilon_2} + \cdots \right)$$

③ Spectral curve geometry.

$\{P(x,y) = 0\} \xrightarrow{x} \mathbb{C}$

$y\,dx$

$$\prod_{a=1}^{r} \left(y - \frac{s_a}{x} \right) = \frac{(-1)^{r+1} \Lambda}{x^{r+1}}$$

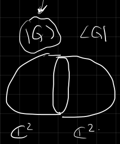
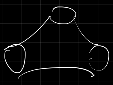
$\Lambda \to 0$

$$\mathcal{C}_{\Lambda=0} : \prod_{a=1}^{r} \left(y - \frac{s_a}{x} \right) = 0$$

Thm Eynard–Orantin TR applied to $\mathcal{C}_{\Lambda=0}$.
(periods on $\mathcal{C}_{\Lambda=0}$) computes the $F_{g,n}$.

In progress) EO-TR on $\mathcal{C}_\Lambda$ computes the $\Phi_{g,n}$.

$\varepsilon_1 + \varepsilon_2 \neq 0$: one has to use a non-commutative version

D-module $\prod_{a=1}^{r} \left(\alpha \partial_{\cancel{x}} - \frac{s_a}{\cancel{x}} \right)$