# Poincaré–Birkhoff–Witt theorems

Algebra, Geometry and Physics Seminar

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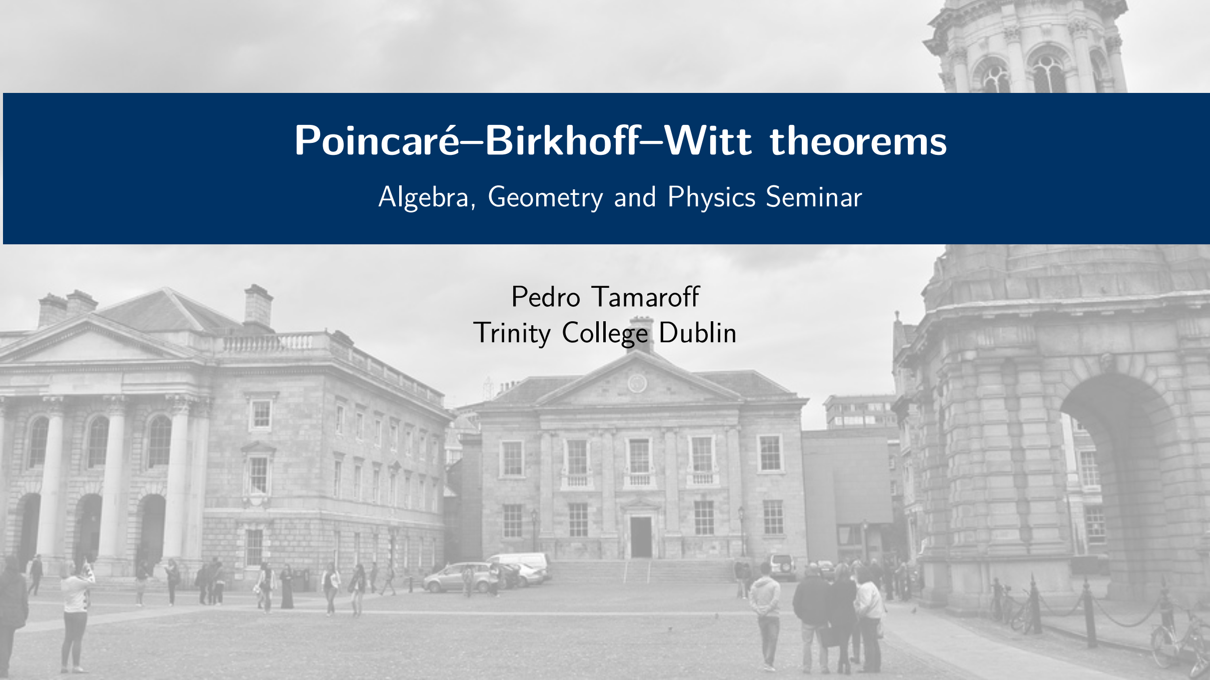

## Motivation and methods



**Theorem (Poincaré–Birkhoff–Witt theorem)**

*Over a field of characteristic zero, the vector space underlying the universal enveloping algebra $U(\mathfrak{g})^\#$ is isomorphic to the symmetric algebra $S(\mathfrak{g}^\#)$, naturally with respect to Lie algebra maps.*

This is an important result in areas such as:

- representation theory,
- homological algebra,
- deformation theory and quantization.

# Motivation and methods



There are many results that fall within the "PBW" umbrella [SW15], which:

- exhibit an algebra as a deformation of another, 'nicer' algebra,
- give us a nice basis of normal monomials for an algebra,
- associate a graded algebra to a non-graded algebra.

We were motivated by the need to produce a formal framework to state and prove such theorems for more general classes of algebras.

## Motivation and methods



*Precursor:* work of Mikhalev and Shestakov [MS14] on varieties of algebras. We were motivated by the need to make precise certain intuitive ideas in their work.

Any functor ${}_S\mathsf{Alg} \longrightarrow {}_T\mathsf{Alg}$ that 'only changes operations' has a left adjoint $A \longmapsto U_S(A)$, so we can attempt to state what a 'PBW-type' theorem is in this case.

*What does it mean that this functor 'does not depend on $A$', exactly?*

For us, this means there is an endofunctor $X$ such that $U_S(A)^\#$ is isomorphic to $X(A^\#)$, naturally with respect to $T$-algebra maps.

## Sketch of the result



We choose the language of endofunctors and monads to give the requisite formal framework.

1. We focus on functors $\psi^*$ that arise by pulling back through a morphism of monads $\psi : F \longrightarrow G$ on some category that is nice enough.

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2. The universal enveloping algebra functor is the pushforward $\psi_!$, given by taking a 'relative tensor product over $F$'.

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2. The universal enveloping algebra functor is the pushforward $\psi_!$, given by taking a 'relative tensor product over $F$'.

3. Our main result shows that this functor is naturally isomorphic to some constant endofunctor on the underlying category if and only if $G$ is a free right $F$-module.

## Consequences



- Organizes 'all at once' computations that were previously done at the level of algebras.
- It allows to use methods coming from rewriting theory and algebraic operads previously mainly used for purposes of homotopical algebra.
- It is fully intrinsic and functorial, and unravels the 'mystery' behind the natural question: what property of the pair (Lie, Ass) makes the PBW theorem work?
- It gives certificates in case the PBW property fails, in the form of homology classes, which can be effectively computed.
- Explains why one cannot expect certain type of results in positive characteristic.

# The Poincaré–Birkhoff–Witt property



### Definition

A monad is an endofunctor $F : \mathsf{C} \to \mathsf{C}$ endowed with an associative composition law $F \circ F \longrightarrow F$ and a unit $\eta : 1 \longrightarrow F$ for it[1].

We were mostly interested in the case when $\mathsf{C} = \mathsf{Vect}$ and $F$ is 'analytic', that is, it takes the form

$$V \longmapsto \bigoplus_{j \geqslant 0} X(j) \otimes_{S_j} V^{\otimes j}$$

for some sequence of representations of the symmetric groups. In this language, elements of the right are 'where we can operate on $V$'.

[1] That is, a monoid in the monoidal category $\mathrm{End}(\mathsf{C})$.

# Monads define algebras



**Definition**

An algebra over a monad $F$ is an object $c \in \mathsf{C}$ along with a structure map $\gamma_c : F(c) \longrightarrow c$ such that $\gamma_c \mu_c = \gamma_c F(\gamma_c)$ and $\gamma_c \eta_c = 1_c$.

In case $F$ is analytic, $F(c) \longrightarrow c$ consists of many maps $X(j) \otimes_{S_j} V^{\otimes j} \longrightarrow V$, so each $x \in X(j)$ defines an equivariant operation $x : V^{\otimes j} \longrightarrow V$.

For example, if $X(j)$ is the trivial representation, then we are considering multi-linear symmetric maps $S^j(V) \longrightarrow V$.

## Morphisms define envelopes



### Definition

If $\psi : F \longrightarrow G$ is a morphism of monads, the pullback functor $\psi^*$ sends a $G$-algebra $(c, \gamma_c)$ to the algebra $(c, \gamma_c \psi_c)$.

The left adjoint $\psi_!$ of $\psi^*$ is called the *universal enveloping algebra functor of* $\psi$. It exists under mild assumptions on C.

For example, if $\psi_V : \mathbb{L}(V) \longrightarrow T(V)$ is the natural inclusion of the free Lie algebra in the tensor algebra, $\psi_!(\mathfrak{g})$ is precisely $U(\mathfrak{g})$.

# How to control this procedure?



The point of a Poincaré–Birkhoff–Witt theorem is to be able to control this abstract process. The only ingredient missing to prove our main result is the following:

**Definition**

A right module over a monad $F$ is another endofunctor $G$ along with a map $G \circ F \longrightarrow G$ compatible with the composition law of $F$.

A right module is *free* if it is isomorphic to one of the form $X \circ F$ where $X$ is an endofunctor and the structure map is $1_X \circ \mu : X \circ F \circ F \longrightarrow X \circ F$.

## The main result



Recall we say the PBW property holds if there is an endofunctor $X$ such that $\psi_!(A)^\#$ is isomorphic to $X(A^\#)$, naturally with respect to algebra maps.

**Theorem (Dotsenko-T.)**

*A map of monads $\psi : F \longrightarrow G$ has the PBW property if and only if it makes $G$ into a free right $F$-module. In this case, the desired functor $X$ is any free right basis of $G$.*

Like the innocent method of counting in two ways, in this case the proof of the result does not matter as much as the fact we have many powerful tools to address whether an object is free or not: this is, more or less, the reason homological algebra exists!

# Sketch of proof



**Proof.**

It is easy to check that for a free $F$-algebra we have $\psi_!(F(d)) = G(d)$, naturally in $d$, so if $\psi_!(c)^\# = X(c^\#)$ we conclude that $G(d) \simeq X(F(d))$ naturally in $c$. Conversely, suppose that $X \circ F = G$. Since $\psi_!(c)$ is computed as the relative composition $G \circ_F c$ —by viewing c as a constant endofunctor— we can simply 'cancel' $F$, leaving $X(c^\#)$ at the end. $\square$

Compare to the case of a morphism of associative algebras $f : A \longrightarrow B$ in which case $f_!(M) = B \otimes_A M$. If $B = V \otimes A$ as an $A$-module, then $B \otimes_A M = V \otimes M$ naturally with respect to maps of $A$-modules. Conversely, $f_!(A) = B$.

# Applications



I will use the following 'homological hammer', which is a weight graded version of the Nakayama lemma for representations and their bar constructions.

**Theorem**

*Let $P$ be a weight graded operad in vector spaces, and let $M$ be a weight graded right module over it. Then $M$ is free if and only if the homology of the one sided bar construction $B(M,P,1)$ vanishes in positive homological degrees. In such case, this module is free with basis $X = H_0(B(M,P,1))$.*

*Main takeaway:* we do not have to guess a Poincaré–Birkhoff–Witt theorem, but rather compute something. The output of that result will tell us what the answer is.

# Classical PBW theorem



There is a map of operads Lie $\longrightarrow$ Ass sending $[x_1, x_2] \longmapsto x_1x_2 - x_2x_1$.

**Theorem**

*The associative operad is free as a right module over the Lie operad with basis given by the endofunctor $V \longmapsto S(V)$.*

**Proof.**

Filter the associative operad using the number of Lie brackets an operation uses (polarize the associative product). The associated graded module is exactly the operad controlling Poisson algebras. As a right module, it is Com $\circ$ Lie, so it is free. By a spectral sequence argument, the associative operad is free with the same basis. $\square$

# A pre-Lie PBW theorem



A class of algebras closely associated to Lie algebras are the pre-Lie algebras, ubiquitous when studying the deformation theory of any class of algebras defined by an algebraic operad.

They are defined by a single operation $x_1 \circ x_2$ whose associator is symmetric in the last two variables. This implies $[x_1, x_2] = x_1 \circ x_2 - x_2 \circ x_1$ is a Lie bracket.

**Theorem**

*The pre-Lie operad is free as a right module over the Lie operad with basis given by the endofunctor $V \longmapsto R(S(V))$.*

Here $R$ is the endofunctor of rooted trees for which no vertex has exactly one child.

# A pre-Lie PBW theorem

### Proof.

Filter the pre-Lie operad using the number of Lie brackets an operation uses. By a result of V. Dotsenko the associated graded module is exactly the operad controlling the $F$-manifold algebras of C. Hertling and Yu. I. Manin. It can be shown it has a basis of tree monomials that is preserved under the action of the Lie operad, so it is free on some sub-basis of it. By a spectral sequence argument, the pre-Lie operad is free with the same basis. □

The fact the basis is given by $R \circ S$ was proved by Dotsenko–Flynn-Connolly using the Koszul complex $K(\mathrm{PreLie}, \mathrm{Lie}, 1)$ to compute bar homology, since Lie is a Koszul operad.

## A question of J.-L. Loday



A dendriform algebra is a vector space $V$ endowed with two operations $x_1 \prec x_2$ and $x_1 \succ x_2$ satisfying three identities that guarantee, in particular, that the operation

$$x_1 \circ x_2 = x_1 \prec x_2 - x_2 \succ x_1$$

is pre-Lie, and the operation $x_1 \cdot x_2 = x_1 \prec x_2 + x_2 \succ x_1$ is Zinbiel up to a term that is quadratic in $x_1 \circ x_2$. In this way, we obtain morphism $\mathrm{PreLie} \longrightarrow \mathrm{Dend}$, and J.-L. Loday wanted to know if a PBW-type theorem existed here.

**Theorem (Dotsenko-T.)**

*The map* $\mathrm{PreLie} \longrightarrow \mathrm{Dend}$ *has the PBW property.*

# A question of J.-L. Loday

The proof also follows a filtration argument through polarization.

**Proof.**

Filter the dendriform operad using the number of pre-Lie brackets an operation uses (polarize the products $\prec, \succ$). The associated graded module is exactly the operad controlling the pre-Poisson algebras of M. Aguiar. This operad has a basis of tree monomials that is preserved under the action of the pre-Lie operad, so it is free on some sub-basis of it. By a spectral sequence argument, the dendriform operad is free with the same basis. $\square$

# Further directions

One of the upshots of having written our work in the language of monads and modules is that we can extrapolate it to more complex settings.

*Differential graded objects.* A natural direction to move towards to is that of 'derived' results: what happens if we allow our objects to be differential graded or allow for homotopy algebras?

*Operads as algebras.* We can also produce an interesting feedback loop if we consider coloured operads: every operad can be made into a pre-Lie algebra, so there is an enveloping operad functor whose inputs are pre-Lie algebras.

# Envelopes of homotopy algebras



We can allow $\alpha : P \longrightarrow Q$ to be a morphism of dg operads. What does it mean to have a PBW property here?

**Definition**

We say $\alpha$ is derived PBW if the natural transformation

$$H(\alpha)_!(HV) \longrightarrow H(\alpha_!(V))$$

is a natural isomorphism for every $P$-algebra $V$.

As the definition shows, the idea is to obtain control on the homology of universal envelopes of 'complicated' algebras through the non-dg envelope $H(\alpha)_!(HV)$.

## Almost-freeness



In this case, the 'correct' notion is that of almost-freeness:

**Definition**

A module is almost free it it admits a filtration whose associated graded module is homotopy equivalent to a free module on a basis of cycles.

The main result we obtained with A. Khoroshkin [KT20] is the following:

# Almost-freeness implies dPBW



**Theorem**

*If $\alpha : P \longrightarrow Q$ makes $Q$ into an almost free right $P$-module with basis of cycles $X$, then $\alpha$ is derived PBW and we have a natural isomorphism*

$$H(\alpha_!(V)) \longrightarrow X(HV).$$

With this at hand, we showed that the $A_\infty$ operad is almost free over the $L_\infty$ operad using techniques coming from homological perturbation theory, producing a unified approach to previous work of V. Baranovsky and J. Moreno-Fernández.

## Some consequences



Concretely, we showed that there is a filtration on $\mathsf{Ass}_\infty$ for which $\operatorname{gr}\mathsf{Ass}_\infty = \mathsf{Poiss}_\infty$ and that $\mathsf{Poiss}_\infty$ is chain homotopy equivalent to $\mathsf{Com}\circ\mathsf{Lie}_\infty$.

- The homology groups of $U(\mathfrak{g})$ for any $L_\infty$-algebra $\mathfrak{g}$ are given by $S(H(\mathfrak{g}))$.
- *Quillen quasi-isomorphism:* for any $L_\infty$-algebra $\mathfrak{g}$ there is a quasi-isomorphism $\mathcal{C}(\mathfrak{g}) \longrightarrow BU(\mathfrak{g})$.
- The models of Baranovsky and of Moreno–Fernández are $A_\infty$-isomorphic and $A_\infty$-quasi-isomorphic to the universal envelope of Lada–Markl.

# Coloured envelopes

There is a map $\mathsf{PreLie}_{\mathbb{N}} \longrightarrow \mathsf{nsOp}$ from the operad controlling weight-graded pre-Lie algebras to the operad controlling ns-operads. An open question is

### Question

Does this morphism satisfy the PBW property?

If so, we could present resolutions of certain operads as envelopes of free dg pre-Lie algebras, opening a door to resolving complicated operads through simple algebras, and attempt to explicitly compute the higher composition maps of certain homotopy operads.

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