Tame geometry and applications

1/ A few theorems

Theorem 1 (Ax) : Let $\mathrm{Exp} = (\exp, \ldots, \exp) : \mathbb{C}^n \longrightarrow (\mathbb{C}^*)^n$.
If $Y \underset{\text{algebraique}}{\subset} \mathbb{C}^n$ then $\overline{\mathrm{Exp}(Y)}^{Zar}$ is a translate of a subtorus of $(\mathbb{C}^\times)^n$.

Rem : This is a functional transcendence result, analogous to the classical Lindemann-Weierstraß theorem :

if $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent, then $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent over $\mathbb{Q}$.

Rem : One can replace $\mathrm{Exp} : \mathbb{C}^n \longrightarrow (\mathbb{C}^\times)^n$ by

$\pi : \mathbb{C}^n \longrightarrow A = \mathbb{C}^n / \mathbb{Z}^{2n}$ Abelian variety (AX)

or

$\pi : D \longrightarrow \Gamma \backslash D$ arithmetic variety (K-Ullmo-Yafaev), $D \subset \mathbb{C}^n$

Theorem 2 (Raynaud):

Let $A = \mathbb{C}^n / \mathbb{Z}^{2n}$ be a complex Abelian variety.

If $C \subseteq A$ irreducible algebraic curve contains infinitely many torsion points

Then C is the translate of an elliptic curve by a torsion point.

Theorem 2' (Faltings; Vojta; McQuillan):

Similar result replacing the subgroup of torsion points of A by the division group of a finitely generated subgroup of A.

Theorem 3 (Cattani – Deligne – Kaplan)

Let $f : X \longrightarrow S$ be a smooth family of complex smooth projective varieties.

The locus $HL(S,d) = \left\{ s \in S \,/\, H^{2d}(X_s, \mathbb{Q}) \text{ contains exceptional Hodge classes} \right\}$

is a countable union of algebraic subvarieties of S (as predicted by the Hodge conjecture).

Nowadays these three results (functional transcendence / diophantine geometry / Hodge theory) can be proven using a common framework:

tame geometry

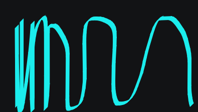
2/ Tame geometry

Grothendieck, Esquisse d'un programme.

Goal: do geometry while discarding wild topological phenomena: Cantor sets, Peano curves, ... but also much more basic examples:

$$\Gamma = \text{graph of} \begin{pmatrix} \mathbb{R}_{>0} \longrightarrow \mathbb{R} \\ x \longmapsto \sin \frac{1}{x} \end{pmatrix}$$

Γ is not tame, for at least 3 reasons:

$\overline{\Gamma} = \Gamma \cup I$ 

a/ $\overline{\Gamma}$ is connected but not arc-connected

b/ $\dim \partial\Gamma = 1 = \dim \Gamma \implies$ no stratification theory

c/ $\Gamma \cap \mathbb{R}$ "not of finite type"

Prototype of tame geometry = Semi-algebraic geometry

Def: $X \subset \mathbb{R}^n$ is semi-algebraic if X is a finite union of $\{x \in \mathbb{R}^n ; f(x) = 0 ; g_i(x) > 0 \ 1 \le i \le k\}$ for some $f, g_1, \ldots, g_k \in \mathbb{R}[X_1, \ldots, X_n]$

Semi-algebraic geometry is too close to algebraic geometry to study a priori non-algebraic phenomena

$\xrightarrow{\text{model Theory}}$ o-minimal structures

Def (structure): a structure expanding $(\mathbb{R}, +, \cdot, <)$ is a collection $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$, S_n set of subsets of $\mathbb{R}^n$ /

1/ algebraic sets of $\mathbb{R}^n$ are in S_n

2/ $S_n \subset \mathcal{P}(\mathbb{R}^n)$ is a boolean subalgebra (i.e. stable under finite $\cup$, finite $\cap$, complement)

3/ $A \in S_p$, $B \in S_q \Rightarrow A \times B \in S_{p+q}$

4/ If $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ linear projection and $A \in S_{n+1}$ then $p(A) \in S_n$.

The elements of S_n are called the $\mathcal{S}$-definable sets of $\mathbb{R}^n$

A function $f : A \to B$ is $\mathcal{S}$-definable if $A, B, \Gamma(f)$ are $\mathcal{S}$-definable.

Rem 1: $\mathcal{S} = \mathbb{R}_{alg}$, where the definable sets are the semi-algebraic sets

4/ = Tarski–Seidenberg

1/ $\Rightarrow \mathbb{R}_{alg} \subset$ (any structure)

Rem 2: $\mathcal{S}_1, \mathcal{S}_2$ two structures $\Rightarrow \mathcal{S}_1 \cap \mathcal{S}_2$ is a structure.

$\Rightarrow$ if $\mathcal{F}$ is a collection of functions $f: \mathbb{R}^n \to \mathbb{R}$ or subsets of $\mathbb{R}^n$, one can define

$\mathbb{R}_{\mathcal{F}}$ = the structure generated by $\mathcal{F}$.

Ex: $\mathbb{R}_{exp}$, $\mathbb{R}_{sin}$

Facts:
- A $\mathcal{S}$-definable $\Rightarrow$ $\bar{A}$, $\mathring{A}$, ∂A $\mathcal{S}$-definable
- $f: A \longrightarrow B$ $\mathcal{S}$-definable $\Rightarrow$ $f(A)$, $f^{-1}(B)$ $\mathcal{S}$-definable
- $\left.\begin{array}{l} f: A \longrightarrow B \\ g: B \longrightarrow C \end{array}\right\}$ $\mathcal{S}$-definable $\Rightarrow$ $g \circ f: A \to C$ is $\mathcal{S}$-definable

Ex: $\bar{A} = \{x \in \mathbb{R}^n ; \forall \varepsilon > 0, \exists y \in A ; \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2\}$

$$= \mathbb{R}^n - P_{n+1,n}\left(\mathbb{R}^{n+1} - P_{2n+1,n+1}(B)\right)$$

where $B = (\mathbb{R}^n \times \mathbb{R} \times A) \cap \{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n ; \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2\}$

Stability under projection $\longleftrightarrow$ elimination of quantifiers

Better to use formulas!

Def (o-minimal structure) a structure $\mathcal{S}$ is o-minimal if s/ any $\mathcal{S}$-definable set in $\mathbb{R}$ is a finite union of points and intervals.

Ex: $\mathbb{R}_{alg}$ is o-minimal ; $\mathbb{R}_{sin}$ is not.

Tame properties: Let $\mathcal{S}$ = any o-minimal structure
definable := $\mathcal{S}$-definable

Thm 1 (Monotonicity) $f : (a,b) \longrightarrow \mathbb{R}$ definable

$\exists\ a = a_0 < a_1 < \dots < a_n = b \ /\ f|(a_i, a_{i+1})$ is $\mathcal{C}^0$ and either constant or strictly monotonous.

Thm 2 (cellular decomposition)

> Let $A_1, \dots, A_k \subset \mathbb{R}^n$ definable.
> There exists a cylindrical definable cellular decomposition of $\mathbb{R}^n$ such that each A_i is a finite union of cells.

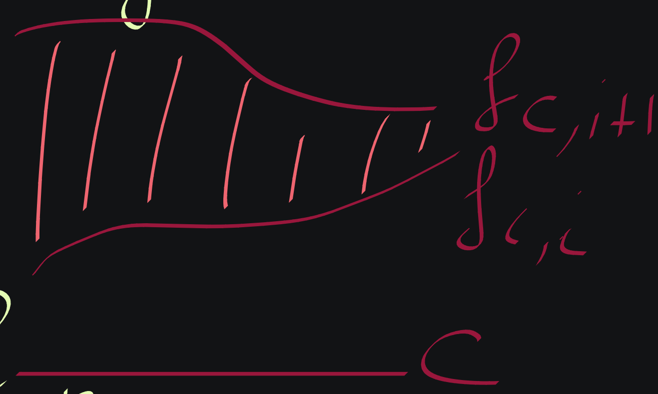
Here: · a CDCD of $\mathbb{R}$ is $a_1 < a_2 < \dots < a_\ell$

$$\text{cells} = \begin{cases} \{a_i\} & 0 < i \leq \ell \\ (a_i, a_{i+1}) & 0 \leq i \leq \ell \quad \begin{pmatrix} a_0 = -\infty \\ a_{\ell+1} = +\infty \end{pmatrix} \end{cases}$$

· a CDCD of $\mathbb{R}^n$ is a CDCD of $\mathbb{R}^{n-1}$

\+ for each cell $C \subset \mathbb{R}^{n-1}$

for definable $f_{C,1} < f_{C,2} < \dots < f_{C,k}$



Cor: A definable $\Longrightarrow$ $|\pi_0(A)| < +\infty$ and any connected component of A is definable.

$\Longrightarrow \dim \partial A < \dim A$

Thm 3 (trivialisation) $f : X \longrightarrow Y$ continuous and definable.

$\exists$ partition of $Y = \coprod_{1 \leq i \leq n} Y_i$ with Y_i definable /

$$\begin{tikzcd}
f^{-1}(Y_i) \arrow[r, "f|_{f^{-1}(Y_i)}"] \arrow[d, "\cong" {rotate=90, anchor=south}, phantom] & Y_i \arrow[d, equal] \\
Y_i \times Z_i \arrow[r] & Y_i
\end{tikzcd}$$

with Z_i definable.

Cor: $A \subset \mathbb{R}^m \times \mathbb{R}^n$ definable $\Rightarrow$ the family $(A_t)_{t \in \mathbb{R}^n}$ of subsets of $\mathbb{R}^m$ takes only a finite # of homeo types.

Examples of o-minimal structures

Ex 1: $\mathbb{R}_{alg}$

Ex 2: $\mathbb{R}_{an} = \mathbb{R}\langle f : [-1,1]^n \longrightarrow \mathbb{R} \text{ real analytic} \rangle$

Ex 3: $\mathbb{R}_{exp}$ (Wilkie) ($x \longmapsto x^{\alpha}$, $x \longmapsto e^{-1/x}$ are $\mathbb{R}_{an,exp}$-definable) α irrational

Ex 4: $\mathbb{R}_{an,exp}$ (Miller - Van den Dries)

Globalization

Def: an $\mathcal{S}$-definable topological space M is a topological space M endowed with a finite atlas of charts $\varphi_i : V_i \longrightarrow U_i \subset \mathbb{R}^n$ /

i) $\forall\, i,j$ $U_{ij} := \varphi_j(V_i \cap V_j)$ is definable

ii) $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : U_{ij} \longrightarrow U_{ji}$ are definable

- a morphism of definable topological space is $f : M \longrightarrow M'$ continuous, definable in the charts.

Ex1: X $\mathbb{R}$-algebraic variety
Then $X(\mathbb{R})$ with usual topology carries a canonical $\mathbb{R}_{alg}$-structure

Ex2: quotients

$X \in \mathcal{S}\text{-Top}$
$R \subset X \times X$ closed definable equivalence relation
When is X/R definable?

Thm (Brumfiel, Van den Dries): If R is definably proper (i.e any one of the $p_i : R \to X$ is proper), then the geometric quotient X/R exists in $\mathcal{S}$-Top.

Thm:
Suppose $\Gamma \curvearrowright X$ definably properly discontinuously.
The choice of an open definable fundamental set $\mathcal{F} \subset X$ for Γ (if it exists) endows ${}_\Gamma X$ with an $\mathcal{S}$-definable structure (which <u>depends</u> on the choice of $\mathcal{F}$).

Ex: $\underline{G}$ semi-simple $\mathbb{R}$-algebraic
$H \subset G := \underline{G}(\mathbb{R})$
compact
$\Gamma < \underline{G}(\mathbb{Q})$ arithmetic

Then $S_{\Gamma, G, H} := {}_\Gamma \backslash G /_H$ has a canonical, functorial $\mathbb{R}_{alg}$-structure.

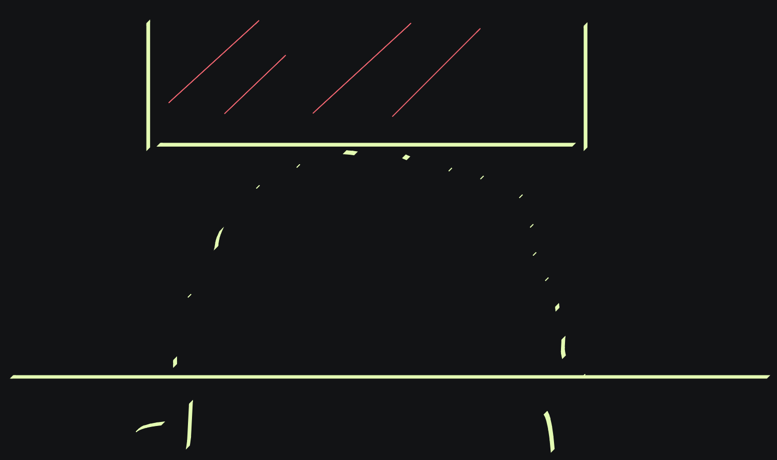
Take for $\mathcal{F}$ a finite union of Siegel sets.

$$G(\mathbb{R}) = N \cdot A \cdot K$$

unipotent split torus maximal compact

$$\mathfrak{F} = \Omega_N \cdot A^+ \cdot K/_{\pm 1}$$

$$SL(2,\mathbb{R}) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



$$\longrightarrow \quad SL(2,\mathbb{Z})\backslash\mathbb{H} \overset{j}{\cong} \mathbb{C}$$

3/ Algebraization

3.1 Diophantine criterion

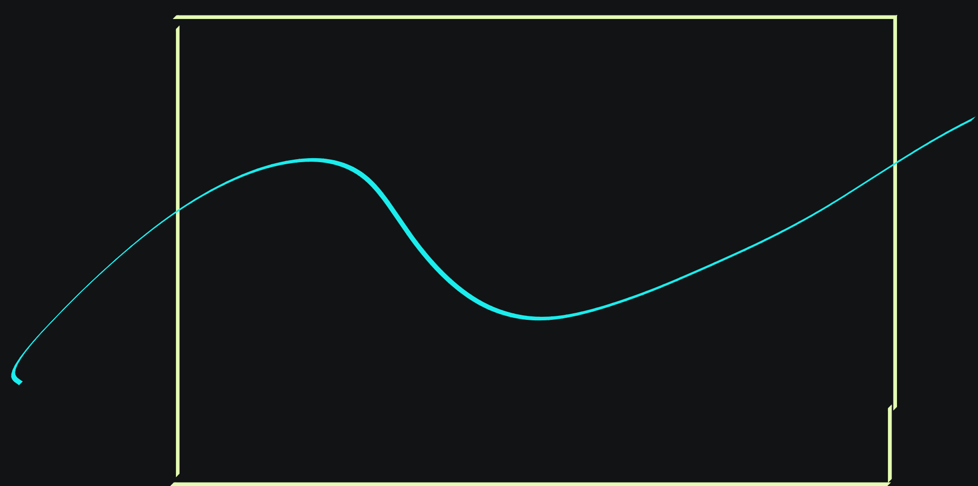
Thm (Pila–Wilkie) $Z \subset \mathbb{R}^n$ $\mathcal{S}$-definable

$Z^{alg} :=$ union of all positive dim. connected semi-algebraic subsets of Z

Then: $\forall \varepsilon > 0$, $\exists\, C(\varepsilon) > 0$ /

$$\#\{x \in (Z \setminus Z^{alg}) \cap \mathbb{Q}^n,\ H(x) \leq T\} < C_\varepsilon T^\varepsilon .$$

Ex:



$C \in \mathbb{R}_{an}$

If $C \cap \mathbb{Q}^2$ is "sufficiently big" then C is real algebraic

3.2 Tame geometry and complex analysis

Motto: the pathologies of complex analysis are not compatible with tame topology.

Lemma: Let $f : \Delta^\times \longrightarrow \mathbb{C}$ be a holomorphic function, definable in some o-minimal structure ($\mathbb{C} \simeq \mathbb{R}^2$).
Then f is meromorphic: 0 is not an essential singularity.

Pf: otherwise, $\{0\} \times \mathbb{C} \subset \partial\Gamma(f)$ by the Great Picard Theorem

$\Rightarrow \dim_{\mathbb{R}} \partial\Gamma(f) = 2 = \dim \Gamma(f)$: contradiction to tameness of f.

Theorem (Remmert-Stein)
Peterzil - Starchenko

definable $\mathbb{C}$-analytic, irreducible

definable $\mathbb{C}$-manifold

$\mathbb{C}$-analytic subset

$$X \subset S \setminus E$$

~~If $\dim X > \dim E$~~ then $\overline{X} \subset S$ is $\mathbb{C}$-analytic, with $\dim \overline{X} = \dim X$

Corollary (Peterzil - Starchenko; o-minimal Chow theorem)

Let $\underset{\substack{\mathbb{C}\text{-analytic,} \\ \text{definable}}}{X} \not\subset \mathbb{C}^n$; then X is algebraic

Rem: one can replace $\mathbb{C}^n$ by any quasi-projective complex variety.

4/ Back to the theorems

The main idea in the proofs of Thm 1, 2, 3 is to prove that some map is tame, i.e definable in some o-minimal structure.

<u>Thm 1</u>: Let $\pi : \mathbb{C}^n \longrightarrow A = \mathbb{C}^n/\Lambda$ the uniformization of an Abelian variety.

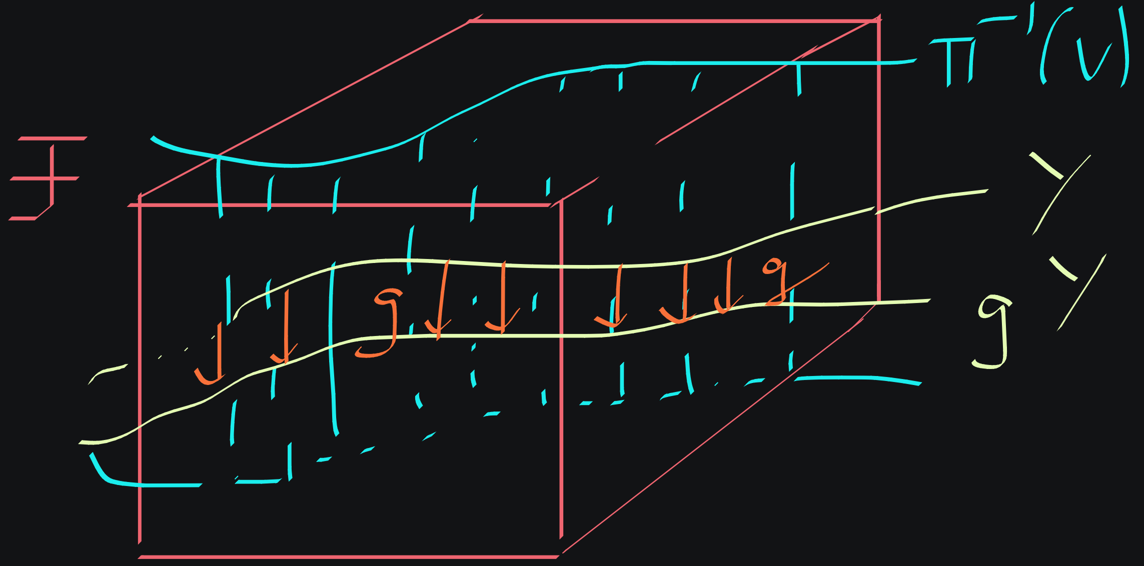
If $Y \subset \mathbb{C}^n$ (algebraic irreducible) then $\overline{\pi(Y)}^{Zar}$ is a translate of an Abelian subvariety

<u>Proof</u> (in the case where A is simple)

- Let $V := \overline{\pi(Y)}^{Zar}$. We want to show that $V = A$ as soon as $Y \neq \{*\}$.
- WLOG: can assume that Y is maximal in $\pi^{-1}(V)$.
- In that case, we show that $\Theta_Y := \mathrm{Stab}_{\mathbb{C}^n} Y$ has positive dimension.

  Then: $V = \overline{\pi(Y)}^{Zar}$ is stable under $\overline{\pi(\Theta_Y)}^{Zar} = A$ thus $V = A$.
- Let us choose $\mathcal{F} \subset \mathbb{C}^g$ a fundamental set for Λ

Then $\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow A$ is $\mathbb{R}_{an}$-definable



Let $\Sigma(Y) := \left\{ g \in \mathbb{C}^n,\ \begin{matrix}(Y+g) \cap \mathcal{F} \neq \emptyset \\ Y+g \subset \pi^{-1}(V)\end{matrix} \right\} \subset \mathbb{C}^n$

$$= \{ g \in \mathbb{C}^n,\ \dim (Y+g) \cap \mathcal{F} \cap \pi^{-1}(V) = \dim Y \}$$

- $\Sigma(Y)$ is $\mathbb{R}_{an}$-definable

- It is enough to show that $\Sigma(Y)$ contains a positive dimensional semi-algebraic set W; by maximality of Y, $Y+W=Y$ hence $W \subset \Theta_Y$, thus $\dim \Theta_Y > 0$.

- Consider $\Sigma(Y) \cap \Lambda = \{ g \in \Lambda,\ Y \cap (g+\mathcal{F}) \neq \emptyset \}$

Exercice: $|\{ g \in \Sigma(Y) \cap \Lambda,\ \underset{\shortparallel}{H(g)} \leq T \}| \geq \frac{T}{2}$

L^∞-norm of g w.r.t. Λ

- By Pila-Wilkie:
$\Sigma(Y)^{alg} \neq \emptyset$.

$\square$

Theorem 2 (Raynaud)

Let $A/\overline{\mathbb{Q}}$ be a simple complex Abelian variety.
Let $V/\overline{\mathbb{Q}} \subsetneq A$ an irreducible algebraic subvariety.
Then V contains only finitely many torsion points.

Proof

- $\pi^{-1}(V) \cap \mathcal{F}$ is $\mathbb{R}_{an}$-definable as $\pi : \mathcal{F} \longrightarrow A$ is.
- By thm 1, $V \neq A \Rightarrow \pi^{-1}(V) \cap \mathcal{F}$ does not contain any positive dimensional semi-algebraic set.
- Pila-Wilkie:

(1) $\forall \varepsilon > 0, \exists C_\varepsilon \,/\, |\{z \in \pi^{-1}(V) \cap \mathcal{F} \cap \Lambda_{\mathbb{Q}} \,/\, H(z) \leq T\}| \leq C_\varepsilon T^\varepsilon$

- If $P \in A$ torsion point with $P = \pi(z)$ for $z \in \mathcal{F} \cap \Lambda_{\mathbb{Q}}$, then $H(z) =$ order P
- Masser:

(2) $\exists c, \rho > 0, \ |Gal(\overline{\mathbb{Q}}/K) \cdot P| \ (= [K(P) : K]) \geq c\, H(z)^\rho$

- (1) + (2) : any torsion point P contained in V has bounded order

$\Longrightarrow$ there are only finitely many such points.

$\square$

Theorem 3 (Cattani – Deligne – Kaplan)

Let $f : X \to S$ be a smooth family of complex smooth projective varieties.

The locus $HL(S,d) = \left\{ s \in S \,/\, H^{2d}(X_s, \mathbb{Q}) \text{ contains exceptional Hodge classes} \right\}$

is a countable union of algebraic subvarieties of S (as predicted by the Hodge conjecture).

Idea of proof (following Bakker – K – Tsimerman 2020)

- The $H^{2d}(X_s, \mathbb{Q})$, $s \in S$, form a $\mathbb{Q}$-local system $\mathbb{V}$ on S
- The associated holomorphic vector bundle $\mathcal{V}$ admits a filtration $F^{\cdot}\mathcal{V}$ (the Hodge filtration: $F^p H^{2d}(X_s, \mathbb{C}) = \bigoplus_{r \geq p} H^{r, 2d-r}(X_s)$)
- This defines a holomorphic period map:

$$\begin{tikzcd} S \arrow[r, "\Phi"] & \Gamma\backslash D = \Gamma\backslash G/H \\ HL(S,d) \arrow[u, hook] \arrow[r] & \coprod_{\text{countable}} \Gamma'\backslash D' \arrow[u, hook, "\text{holomorphic}"'] \end{tikzcd}$$

classifying the variation of $F^{\cdot}$ as s moves.

$$\begin{tikzcd} S \arrow[r, "f"] & \Gamma\backslash D \\ Hl(S,d) \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner"] & \coprod \Gamma'\backslash D' \arrow[u, hook] \end{tikzcd}$$

S is $\mathbb{C}$-algebraic hence has a natural $\mathbb{R}_{alg}$-structure

$\Gamma\backslash D$ has a natural $\mathbb{R}_{alg}$-structure / $\Gamma'\backslash D' \hookrightarrow \Gamma\backslash D$ $\mathbb{R}_{alg}$-definable

<u>Thm</u> (B-K-T): $f: S \longrightarrow \Gamma\backslash D$ is $\mathbb{R}_{an,exp}$-definable.

Hence: each $f^{-1}(\Gamma'\backslash D') \subset S$ is $\mathbb{C}$-analytic and $\mathbb{R}_{an,exp}$-definable

$$\underset{\substack{\text{o-minimal} \\ \text{Chow}}}{\Longrightarrow} \quad f^{-1}(\Gamma'\backslash D') \underset{\text{algebraic}}{\subset} S$$

$\square$