$\mathcal{W}_\infty$ and integrability

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What are W-algebras?

- *ugly cousins of Virasoro algebra*
- *very natural* generalization of Virasoro algebra
- rich algebraic structure and representation theory
- appear in many areas of mathematical physics
- they have very different descriptions,
  e.g. VOA description (CFT) vs Yangian description (integrability)
- in certain sense one particular example is a continuation of *harmonic oscillator* $\rightsquigarrow$ *2d chiral free boson* $\rightsquigarrow$ $\mathcal{W}_{1+\infty}$ (natural numbers $\rightsquigarrow$ partitions $\rightsquigarrow$ 3d/plane partitions partitions)

Examples of areas where W-algebras show up:

- integrable hierarchies of PDE (KdV/KP) $\rightsquigarrow$ $\mathcal{W}$ is quant. KP
- (old) matrix models
- quantum Hall effect
- topological strings
- 4d $\mathcal{N} = 2$ instanton partition functions (AGT 2009)
- AdS_3/CFT_2 with higher spins (Gaberdiel/Gopakumar 2010)
- 4d $\mathcal{N} = 2$ SCFTs (BLLPRvR 2013)
- twisted M-theory (Costello 2016)
- KW-twisted $\mathcal{N} = 4$ SYM at junction of three codimension 1 defects (Gaiotto, Rapčák 2017)
- mathematics: geometric representation theory (equivariant cohomology of various moduli spaces)

Zamolodchikov $\mathcal{W}_3$ algebra

$\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

and additional spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

To close the algebra we need to find the OPE of W with itself consistent with associativity (Jacobi, crossing symmetry...).

The result:

$$W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right) + reg.$$

Λ is a quasiprimary 'composite' (spin 4) field,

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).$$

The algebra is non-linear $\rightsquigarrow$ not a Lie algebra in the usual sense

- the representation theory significantly simplifies if one studies *families* of algebras rather than isolated algebras
- one such family are the algebras $W_N[c]$ which are generated by spins $2, 3, \ldots, N$ [cousins of Virasoro algebra]
- there are various ways of constructing algebras of this family:
  1. direct OPE bootstrap (at lower N)
  2. Drinfeld-Sokolov reduction (Hamiltonian reduction from affine Lie algebra $\widehat{\mathfrak{sl}}(N)$, either classical or quantum BRST)
  3. Goddard-Kent-Olive coset construction

$$\frac{\mathfrak{su}(n)_k \times \mathfrak{su}(n)_1}{\mathfrak{su}(n)_{k+1}} \simeq \frac{\mathfrak{u}(k+1)_n}{\mathfrak{u}(k)_n \times \mathfrak{u}(1)}$$

  4. free field representations (Miura transformation) - more later

But the OPEs are still not very encouraging:

$$
\begin{aligned}
U_3(z)U_4(w) \quad \sim \quad & \frac{1}{(z-w)^7}\Big(\frac{1}{2}\alpha(n-3)(n-2)(n-1)n\left(4\alpha^2\left(\alpha^2(n(5n-9)+1)-3n+4\right)+1\right)\mathbb{1}\Big) \\
& +\frac{1}{(z-w)^6}\Big(\frac{1}{6}(n-3)(n-2)(n-1)\left(6\alpha^4 n(2n-3)+\alpha^2(10-9n)+1\right)U_1(w)\Big) \\
& +\frac{1}{(z-w)^5}\Big(-\alpha(n-3)(n-2)(n-1)\left(-4\alpha^2+3\alpha^2 n-1\right)(U_1U_1)(w) \\
& +\alpha(n-3)(n-2)\left(4\alpha^2 n^2-4\alpha^2 n-n-2\right)U_2(w) \\
& -\frac{1}{2}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1'(w)\Big) \\
& +\frac{1}{(z-w)^4}\Big(-\alpha(n-3)(n-2)(n-1)\left(\alpha^2(3n-4)-1\right)(U_1'U_1)(w) \\
& -\frac{1}{2}(n-3)(n-2)\left(2\alpha^2(n-1)-1\right)(U_1U_2)(w) \\
& +(n-3)\left(\alpha^2\left(n^2+2\right)-3\right)U_3(w) \\
& -\frac{1}{4}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1''(w) \\
& +\alpha(n-3)(n-2)\left(\alpha^2(n-1)n-1\right)U_2'(w)\Big) \\
& +\cdots
\end{aligned}
$$

Gaberdiel/Gopakumar triality

- one can make one step further and consider algebra $\mathcal{W}_\infty$ generated by currents of spin $2, 3, \ldots$ [or $\mathcal{W}_{1+\infty}$ if we include spin 1]
- the resulting algebra depends rationally on two parameters, $\mathcal{W}_\infty[c, \lambda] \quad c, \lambda \in \mathbb{C}$ (GG, Linshaw)
- this is analogous to the higher spin / fuzzy sphere algebra

$$\frac{\mathcal{U}(\mathfrak{so}(3))}{\langle X^2 + Y^2 + Z^2 - (\lambda^2 - 1) \rangle}$$

  interpolating between all algebras of $N \times N$ matrices
- to get $W_N[c]$ from $\mathcal{W}_\infty[c, \lambda]$ we put $\lambda = N \in \mathbb{N}$ and quotient out the ideal generated by spins $N + 1$ and higher

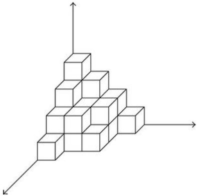
- an important observation (Gabediel/Gopakumar): given c and λ, we can actually find two other values of λ with identical OPEs, $\mathcal{W}_\infty[c,\lambda_1] \simeq \mathcal{W}_\infty[c,\lambda_2] \simeq \mathcal{W}_\infty[c,\lambda_3]$

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=0, \qquad c=(\lambda_1-1)(\lambda_2-1)(\lambda_3-1)$$

- this redundant parametrization manifests the **triality** symmetry $\mathcal{S}_3$ permuting λ_j
- this discrete symmetry controls many aspects of representation theory of $\mathcal{W}_\infty$
- e.g. MacMahon function as vacuum character of the algebra (*enumerating all local fields in the algebra*)

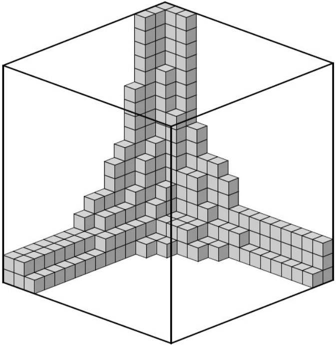
$$\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n}=1+q+3q^2+6q^3+13q^4+24q^5+48q^6+\cdots$$

- The same generating function is well-known to count the plane partitions (3d Young diagrams)



- triality acts by permuting the coordinate axes
- restriction to $\mathcal{W}_N$ corresponds to max N boxes in one of the directions (the simplest truncations of $\mathcal{W}_\infty$)

- the box counting can be generalized to degenerate primaries (not only vacuum rep) by allowing 2d Young diagram asymptotics



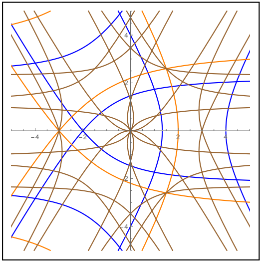
- counting exactly as in topological vertex $\rightsquigarrow$ topological vertex can be interpreted as being a character of (maximally) degenerate $\mathcal{W}_{1+\infty}$ representations

Truncations, minimal models

- for $\lambda_3 = N$ we have $\mathcal{W}_\infty \to \mathcal{W}_N$ (i.e. $\lambda_3 = 2$ is Virasoro)
- more general truncations: the vacuum rep has a singular vector at level $(N_1+1)(N_2+1)(N_3+1)$ if

$$\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 1$$

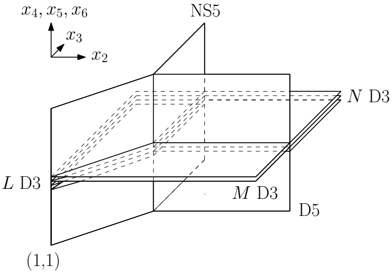
- imposing 2 conditions: minimal models



- example: $Y_{k,k+1,0}$ gives k-th unitary minimal models of $\mathcal{W}_N$

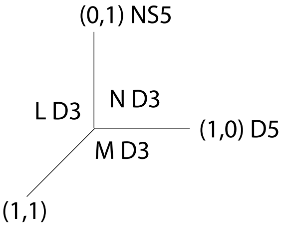
Brane construction (Gaiotto-Rapčák)

simple truncations of $\mathcal{W}_\infty$ - $Y_{N_1 N_2 N_3}$ algebras

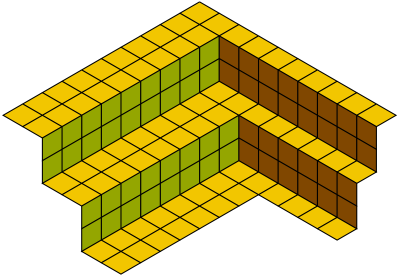


$\times \underset{x_0, x_1}{C} \times \underset{x_7, x_8, x_9}{R^3}$

$$Y_{LMN}[\Psi] = \frac{W^{DS}_{N-M}[U(N|L)_\Psi]}{U(M|L)_{\Psi-1}}$$



- the combinatorics of box counting generalizes also to double truncations (minimal models) - periodic plane partitions / tilings of cylinder
- example: Ising model $c = \frac{1}{2}$ has $\lambda = (2, \frac{2}{3}, -\frac{1}{2})$ and is $Y_{002} \cap Y_{210}$



- adding boxes to this configuration (respecting periodicity) reproduces all characters of W_N minimal models (these are usually written as sums of affine Weyl groups / theta functions...)

Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$
\begin{aligned}
0 &= [e_{j+3}, e_k] - 3\,[e_{j+2}, e_{k+1}] + 3\,[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad +\sigma_2\,[e_{j+1}, e_k] - \sigma_2\,[e_j, e_{k+1}] - \sigma_3\,\{e_j, e_k\} \\
0 &= [f_{j+3}, f_k] - 3\,[f_{j+2}, f_{k+1}] + 3\,[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\
&\quad +\sigma_2\,[f_{j+1}, f_k] - \sigma_2\,[f_j, f_{k+1}] + \sigma_3\,\{f_j, f_k\} \\
0 &= [\psi_{j+3}, e_k] - 3\,[\psi_{j+2}, e_{k+1}] + 3\,[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, e_k] - \sigma_2\,[\psi_j, e_{k+1}] - \sigma_3\,\{\psi_j, e_k\} \\
0 &= [\psi_{j+3}, f_k] - 3\,[\psi_{j+2}, f_{k+1}] + 3\,[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, f_k] - \sigma_2\,[\psi_j, f_{k+1}] + \sigma_3\,\{\psi_j, f_k\} \\
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k]
\end{aligned}
$$

'initial/boundary conditions'

$$[\psi_0, e_j] = 0, \qquad [\psi_1, e_j] = 0, \qquad [\psi_2, e_j] = 2e_j,$$
$$[\psi_0, f_j] = 0, \qquad [\psi_1, f_j] = 0, \qquad [\psi_2, f_j] = -2f_j$$

and finally the Serre relations

$$0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]], \quad 0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]].$$

Parameters $h_1, h_2, h_3 \in \mathbb{C}$ constrained by $h_1 + h_2 + h_3 = 0$ and

$$\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3$$
$$\sigma_3 = h_1 h_2 h_3.$$

We have both commutators and anticommutators in defining quadratic relations (but no $\mathbb{Z}_2$ grading) - for $\sigma_3 \neq 0$ not a Lie (super)-algebra.

Introducing generating functions

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

the first set of formulas above (almost) simplify to

$$\begin{aligned} e(u)e(v) &= \varphi(u-v)e(v)e(u), \quad & f(u)f(v) &= \varphi(v-u)f(v)f(u), \\ \psi(u)e(v) &= \varphi(u-v)e(v)\psi(u), \quad & \psi(u)f(v) &= \varphi(v-u)f(v)\psi(u) \end{aligned}$$

with rational structure function (scattering phase in BAE)

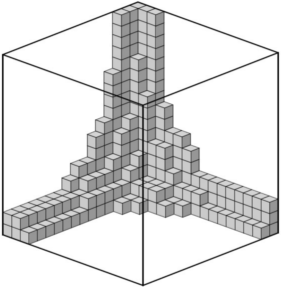
$$\boxed{\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}}$$

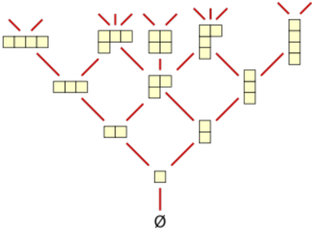
The representation theory of the algebra is much simpler in this Yangian formulation and it is controlled by this single function the generating functions $\psi(u), e(u)$ and $f(u)$ typically act as

$$\psi(u)\,|\Lambda\rangle = \psi_0(u) \prod_{\square\in\Lambda} \varphi(u - h_\square)\,|\Lambda\rangle$$

$$e(u)\,|\Lambda\rangle = \sum_{\square\in\Lambda^+} \frac{E(\Lambda\to\Lambda+\square)}{u-h_\square}\,|\Lambda+\square\rangle$$

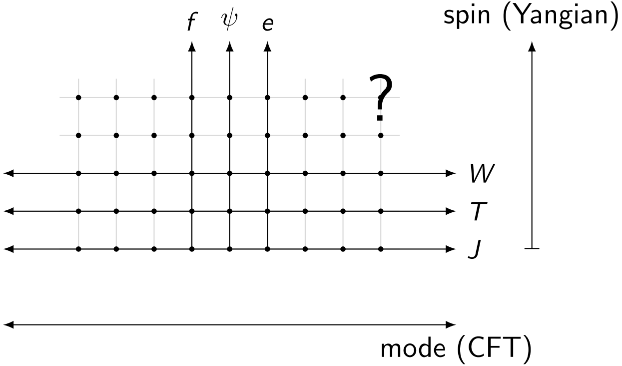
where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions,...) and where $h_\square = \sum_j h_j x_j(\square)$ is the weighted geometric position of the box.





Two different descriptions of the algebra:

- usual CFT point of view with local fields $J(z), T(z), W(z), \ldots$ with increasingly complicated OPE as we go to higher spins
- Yangian point of view (Arbesfeld-Schiffmann-Tsymbaliuk) where all the spins are included in the generating functions $\psi(u), e(u)$ and $f(u)$ but accessing higher mode numbers is difficult



How to connect these two descriptions?

- the parameters can be identified as $\lambda_j \sim \frac{1}{h_j}$
- the generators with low spin and mode can be identified

$$\psi_2 = 2L_0, \quad e_0 = J_{-1}, \quad f_0 = -J_{+1}$$

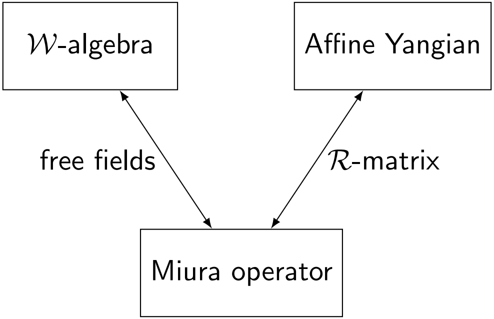
$$\psi_3 = (W_3 + \ldots)_0 + \sigma_3 \sum_{m>0} (3m-1) J_{-m} J_m$$

  (cut & join operator, *not a zero mode of a local field*, Hilbert transform $\rightsquigarrow$ Benjamin-Ono equation)

- this is sufficient to find the map spin by spin, but what is the more conceptual way to understand the map?
- Negut: closed form of the map $\mathcal{W} \to \mathcal{Y}$ using shuffle algebra

Miura operator

very powerful (free field rep, coproduct, integrability)



- consider the following factorization of N-th order differential operator

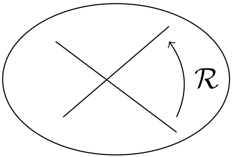
$$(\partial + \partial\phi_1(z)) \cdots (\partial + \partial\phi_N(z)) = \sum_{j=0}^{N} U_j(z) \partial^{N-j}$$

  with N commuting free fields $\partial\phi_j(z)\partial\phi_k(w) \sim \delta_{jk}(z-w)^{-2}$
- OPEs of U_j generate $\mathcal{W}_N$ and furthermore are quadratic $\rightsquigarrow$ free field representations of $\mathcal{W}_\infty$
- $\mathcal{W}_N \leftrightarrow$ quantization of the space of N-th order differential operators as Hamiltonian system (KdV_N and KP hierarchies)

Miura transformation and $\mathcal{R}$-matrix

- the embedding of $\mathcal{W}_N$ in the bosonic Fock space $\mathcal{F}^{\otimes N}$ depends on the way we order the fields
- $\mathcal{R}$-matrix as transformation (intertwiner) between two embeddings, $\mathcal{R} : \mathcal{F}^{\otimes 2} \to \mathcal{F}^{\otimes 2}$

$$(\partial + \partial\phi_1)(\partial + \partial\phi_2) = \mathcal{R}^{-1}(\partial + \partial\phi_2)(\partial + \partial\phi_1)\mathcal{R}$$

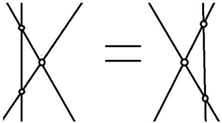


- actually have three different elementary Miura corresponding to three different asymptotics, always of the dressing form

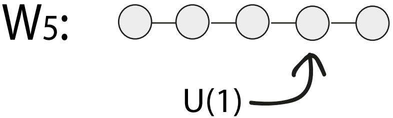
$$(\partial + \partial\phi(z)) = e^{-\phi(z)}\partial^{\#}e^{\phi(z)}$$

- $\mathcal{R}$ defined in this way satisfies the Yang-Baxter equation (two ways of reordering $321 \to 123$)

$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) = \\ = \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2)$$



- the spectral parameter u - the global $U(1)$ charge
- $\mathcal{R}$-matrix satisfying YBE $\rightsquigarrow$ apply the algebraic Bethe ansatz



- spin chain of length $N \rightsquigarrow \mathcal{W}_N$ algebra (level N Yangian)

- consider an *auxiliary* Fock space $\mathcal{F}_A$ and a *quantum* space $\mathcal{F}_Q \equiv \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_N$
- we associate to this the monodromy matrix $\mathcal{T}_{AQ} : \mathcal{F}_A \otimes \mathcal{F}_Q \to \mathcal{F}_A \otimes \mathcal{F}_Q$ defined as

$$\mathcal{T}_{AQ} = \mathcal{R}_{A1}\mathcal{R}_{A2} \cdots \mathcal{R}_{AN}$$

- if the individual $\mathcal{R}$-matrices satisfies the YBE, $\mathcal{T}$ will also satisfy YBE with respect to two auxiliary spaces A and B

$$\mathcal{R}_{AB}\mathcal{T}_A\mathcal{T}_B = \mathcal{T}_B\mathcal{T}_A\mathcal{R}_{AB}$$

- the algebra of matrix elements of $\mathcal{T}$ satisfying this equation is the Yangian (in RTT presentation)

- in our situation the Fock-Fock $\mathcal{R}$-matrix controlling the commutation relations of Yangian has rows and columns labeled by Young diagrams ($\rightsquigarrow$ infinite number of generators)
- we can however restrict to simple matrix elements, i.e. upper left corner of the $\mathcal{R}$-matrix

$$\mathcal{H} = \langle 0|_A \mathcal{T} |0\rangle_A , \quad \mathcal{E} = \langle 0|_A \mathcal{T} |1\rangle_A , \quad \mathcal{F} = \langle 1|_A \mathcal{T} |0\rangle_A$$

- the YBE now implies relations between these operators like

$$0 = [\mathcal{H}(u), \mathcal{H}(v)]$$

(infinite set of commuting Hamiltonians) or

$$(u - v + h_3)\mathcal{H}(u)\mathcal{E}(v) = (u - v)\mathcal{E}(v)\mathcal{H}(u) + h_3\mathcal{H}(v)\mathcal{E}(u)$$

(ladder operators)

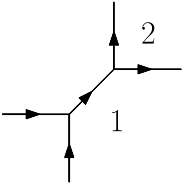
- these generating functions can be related to AST Yangian

$$\psi(u) = \frac{u+\sigma_3\psi_0}{u}\frac{\mathcal{H}(u+h_1)\mathcal{H}(u+h_2)}{\mathcal{H}(u)\mathcal{H}(u+h_1+h_2)}$$
$$e(u) = h_3^{-1}\mathcal{H}(u)^{-1}\mathcal{E}(u)$$
$$f(u) = -h_3^{-1}\mathcal{F}(u)\mathcal{H}(u)^{-1}$$

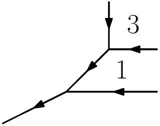
- using $\mathcal{R}$-matrix one can find these Yangian generators systematically following the algorithm of QISM (the only input is the Miura operator $\partial + J(z)$)

Generalizations

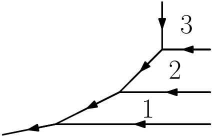
- gluing: generalizations to more complicated VOAs obtained by gluing $\mathcal{W}_\infty$ as basic building block as in topological vertex formalism for toric Calabi-Yaus; this is particularly natural from Gaiotto-Rapcak point of view (TP & Rapcak)



$\mathcal{N} = 2$ SCA

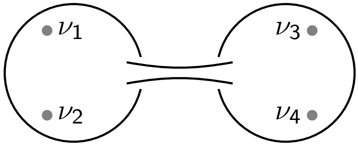


Bershadsky-Polyakov $W_3^{(2)}$



$\widehat{\mathfrak{gl}}(3)$

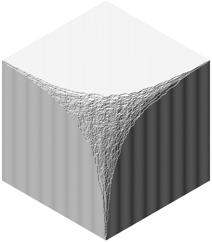
- matrix $\mathcal{W}_\infty$: many properties are unchanged if we replace $\widehat{\mathfrak{gl}}(1) \rightsquigarrow \widehat{\mathfrak{gl}}(M)$ (Eberhardt & TP), is there higher integrability such as Zamolodchikov's tetrahedron equation?
- Grassmannians: there exists three (conjecturally four) parametric generalization of $\mathcal{W}_\infty$ that allows construction of even larger class of VOAs including the unitary Grassmannians, Lagrangian (ortho-unitary) Grassmannians, $\mathcal{N} = 4$ SCA, $\mathfrak{d}(2, 1, \alpha)$ and conjecturally has pentality symmetry $\mathcal{S}_5$ (Eberhardt & TP)

$\mathrm{Gr}(\nu_1, \nu_2, \nu_3, \nu_4) =$ 

Questions

Many possible directions (work in progress...)

- Bethe equations (BLZ, Litvinov), interpolating classes of commuting ILW Hamiltonians (Yangian-local BLZ)
- elliptic Calogero-Moser models and modularity
- ODE/IM correspondence, quantum spectral curve and WKB, B-model of topological string, topological recursion



- construction of the 4-parametric Grassmannian algebra, maybe ODE/IM can give hints
- string or SYM realizations - M-theory interpretation of many elements (Gaiotto-Rapcak), can this lead to some interesting predictions not expected from CFT/integrability point of view?

Thank you!

How does the $\mathcal{R}$-matrix look like?

- a fermionic form obtained by A. Smirnov by studying the large N limit of $\mathfrak{gl}(N)$ $\mathcal{R}$-matrix - complicated
- a bosonic formula is not known so far, but results of Nazarov-Sklyanin can be interpreted as matrix elements of $\mathcal{R}$-matrix in mixed representation where the two bosonic Fock spaces are associated to different asymptotic directions

$$\mathcal{R}(u) = \mathbb{1} - \frac{1}{u}\sum_{j>0} a_{-j}a_j + \frac{1}{2!u(u+1)}\sum_{j,k>0}(a_{-j}a_{-k}+a_{-j-k})(a_j a_k + a_{j+k})$$
$$- \frac{1}{3!u(u+1)(u+2)}\sum_{j,k,l>0}(a_{-j}a_{-k}a_{-l} + a_{-j-k}a_{-l} + a_{-j-l}a_{-k}$$
$$+ a_{-k-l}a_{-j} + 2a_{-j-k-l})(\ldots) + \ldots$$

with $a_j \equiv a_j^{(1)} - a_j^{(2)}$.

Truncations and conjectural 4-parametric algebra

The truncation curves are best parametrized if we introduce additional *two* parameters $\nu_4 = 1$ and $\nu_5 = -1$. The central charge becomes

$$c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)(\lambda_4 - 1)(\lambda_5 - 1)$$

where $\lambda_j = (\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5)/\nu_j$ and so

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} + \frac{1}{\lambda_5} = 1$$

We find a truncation at level

$$(N_1 + 1)(N_2 + 1)(N_3 + 1)(N_4 + 1)(N_5 + 1)$$

if the parameters λ_j satisfy

$$\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} + \frac{N_4}{\lambda_4} + \frac{N_5}{\lambda_5} = 1$$

just like in $\mathcal{W}_{1+\infty}$.