Intrinsic Mirror Symmetry

(Joint work with Bernd Siebert)

Enumerative geometry (19th century)

Count geometric objects,

e.g. Suppose given 5 lines in the plane. How many conics are tangent all 5 lines?

e.g. Given a cubic surface, i.e., a surface defined by a cubic polynomial, how many straight lines are contained in the surface?

Cayley-Salmon (1849) Every non-singular cubic surface contains precisely

27 lines (over $\mathbb{C}$).

E.g.
Suppose given a quintic 3-fold in

$$\mathbb{C}\mathbb{P}^4 = \left(\mathbb{C}^5 \setminus \{0\}\right) / \mathbb{C}^*$$

$$(x_0, \dots, x_4) \sim (\lambda x_0, \dots, \lambda x_4)$$

How many lines does such a quintic contain?

(H. Schubert) 2875 for the general quintic.
19th

How many conics does such a quintic contain?

(S. Katz 1986) 609,250

How many twisted cubics does a general quintic contain?

(Here we want count images of

(injective) maps

$$\varphi : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^4$$

with (u,v) above $\mathbb{CP}^1$ and $(x_0, \dots, x_4)$ above $\mathbb{CP}^4$

given by homog. polynomials of degree 3.

$$\varphi(u,v) = (f_0(u,v), \dots, f_4(u,v))$$

where $f_0, \dots, f_4$ are homog. poly's of deg. 3 with no common zero.

Ellingsrud + Strømme (1990) $\sim 3 \times 10^8$

More generally, can we compute the number N_d which is, roughly, defined as the number of images of maps $\varphi : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^4$ given by homog. poly's of degree d (with no common zeros) contained in the quintic.

Mirror Symmetry.

String theory : 1989.

Fundamental observation:

We want to have space-time being 10-dimensional, of the form

$$\mathbb{R}^{1,3} \times X$$

↳ very small six-dim'l manifold.

X should be a Calabi-Yau manifold, i.e., should be a compact 3-dim'l complex manifold with a Ricci-flat metric.

Yau's Theorem tells us that X has a Ricci-flat metric if it is a non-singular projective variety with a nowhere vanishing holomorphic 3-form.

e.g. the quintic 3-fold.

<u>Suggestion</u>! C-Y manifolds should come in pairs X, $\check{X}$, with

$$\chi_{top}(X) = -\chi_{top}(\check{X}).$$

More precisely, we have

$$h^{p,q}(X) = h^{3-p,q}(\check{X})$$

(equality of Hodge numbers)

Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

<u>Example</u>! Greene-Plesser

Candelas et al

Mirror to quintic

$$X = Z(x_0^5 + \cdots + x_4^5) \subseteq \mathbb{CP}^4$$

(Fermat quintic)

$\mathbb{Z}_5^5 \circlearrowright \mathbb{CP}^4$ $\quad (a_0, \dots, a_4) \in \mathbb{Z}_5^5$

acts by

$$(x_0, \dots, x_4) \mapsto (\xi^{a_0} x_0, \dots, \xi^{a_4} x_4)$$

where $\xi = e^{2\pi i/5}$

This group also acts on X

Let $G = \{(a_0, \dots, a_4) \mid \sum a_i = 0\} \subseteq \mathbb{Z}_5^5$

$\cong \mathbb{Z}_5^4$

G also acts on X, and obtain

a very singular quotient X/G

$\exists$ a resolution of singularities

$$\check{X} \to X/G \quad \text{such that}$$

$\check{X}$ is also C-Y.

$h^{1,1}(X) = 1, \quad h^{1,2}(X) = 101$

$h^{1,1}(\check{X}) = 101, \quad h^{1,2}(\check{X}) = 1$

Candelas, de la Ossa, Green, and Parkes (1990)

Used predictions from string theory to give predictions for all the numbers N_d in terms of "period integrals" on $\check{X}$, i.e., integrals of the form

$$\int_\alpha \check{\Omega} \quad \text{where } \check{\Omega} \text{ is}$$

the nowhere vanishing holomorphic 3-form on $\check{X}$ and $\alpha \in H_3(\check{X}, \mathbb{Z})$

<u>Givental:</u> 1996, proved the Candelas et al predictions.

<u>Batyrev, (Batyrev-Borisov) '92, ('94):</u>

Gave a broad construction of mirror pairs for hypersurfaces (complete intersections) in toric varieties.

$\sim 4 \times 10^8$

Strominger-Yau-Zaslow conjecture (1996)

Gave a geometric proposal to explain the relationship between mirror pairs in terms of "special Lagrangian torus fibrations."

Basic questions:

① Is there a general construction of mirror pairs?

② Is there a conceptual explanation for the observed equalities between curve counts and period integrals?

Will give a proposal for ① today

(w/ B. Siebert, "Intrinsic mirror symmetry," 2019.)

<u>Our context:</u>

We fix a <u>log Calabi-Yau pair</u>

(X, D) where

- X is a non-singular projective variety (n-dim'l)

Locally looks like $\prod x_i = 0$

- D is a <u>reduced normal crossing</u> divisor with $K_X + D = 0$

(i.e., $\exists$ a nowhere vanishing holomorphic n-form on $X \setminus D$ with simple poles along components of D.)

Will construct something which should be the mirror to $X \setminus D$.

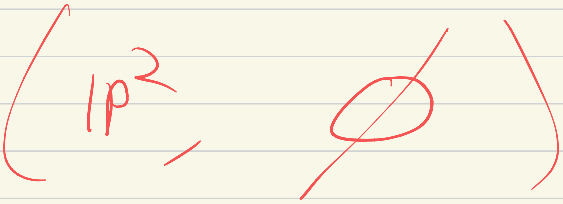
Write $D = D_1 + \cdots + D_s$ the decomposition into irreducible components.

<u>Assume</u>: For $I \subseteq \{1, \dots, s\}$

$$D_I := \bigcap_{i \in I} D_i \quad \text{is connected.}$$

e.g.

Good



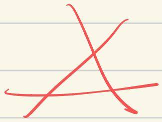
Bad

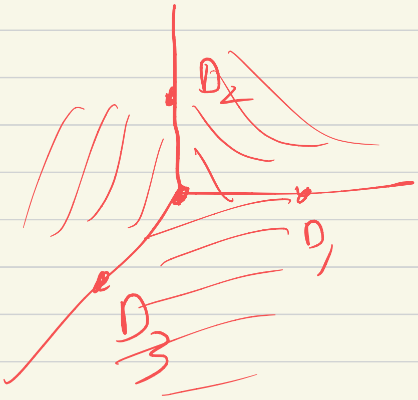
Will require some combinatorial construction.

We build the <u>dual complex</u> of (X, D) a cone complex in the $\mathbb{R}$-vector space with basis $D_1, \dots, D_s$

$$\mathcal{P} := \left\{ \sum_{i\in I} \mathbb{R}_{\geq 0} D_i \;\middle|\; I \subseteq \{1, \dots, s\},\ D_I \neq \emptyset \right\}$$

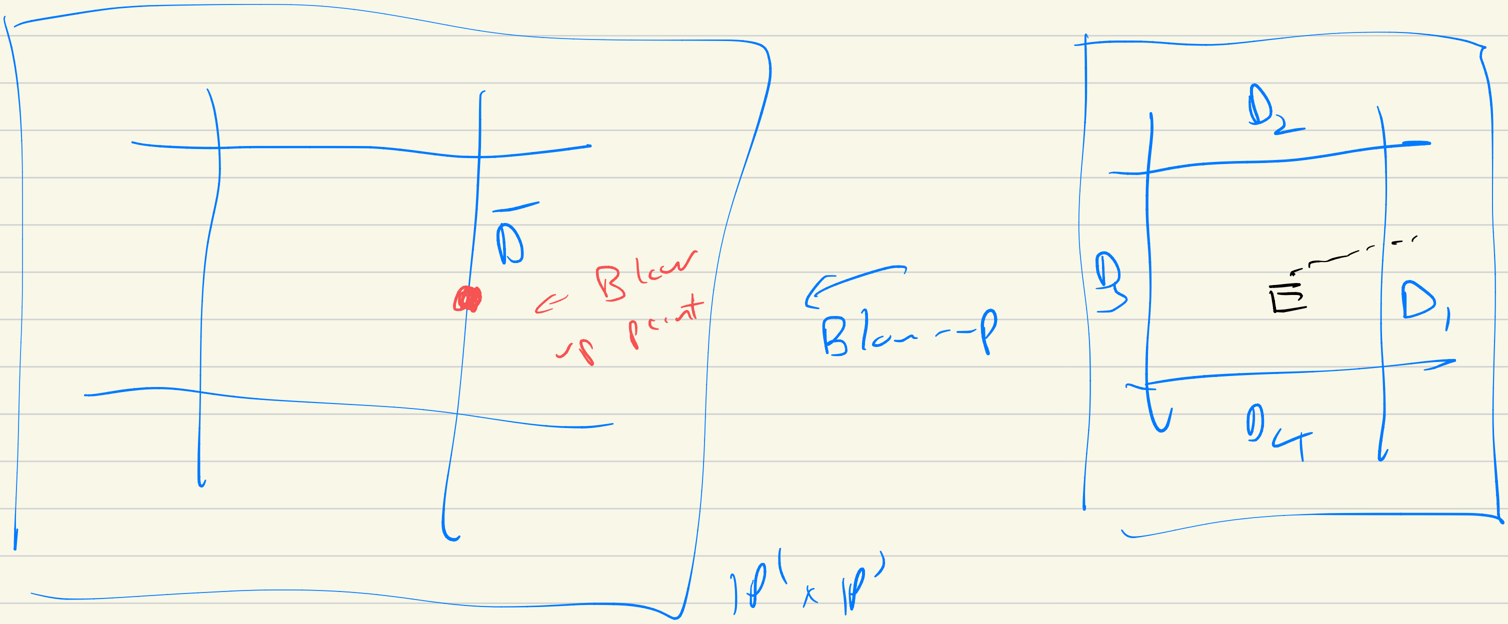
$$B = \bigcup_{\sigma\in\mathcal{P}} \sigma$$

e.g. $\left(\mathbb{P}^2,\right.$  $\left.\right)$



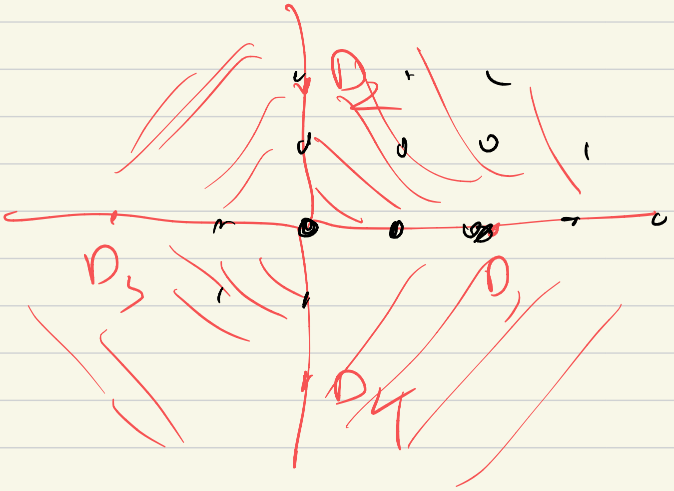
Example: Start with

$$\left(\mathbb{P}^1\times\mathbb{P}^1,\quad \bar{D} = (\{0,\infty\}\times\mathbb{P}^1) \cup (\mathbb{P}^1\times\{0,\infty\})\right)$$



Take D to be the strict transform of $\overline{D}$.

(B, D)



$$B(\mathbb{Z}) = \{ \textstyle\sum a_i D_i \in B \mid a_i \in \mathbb{Z}_{\geq 0} \}.$$

A point of $B(\mathbb{Z})$ records contact orders of maps $f : C \longrightarrow X$, C a curve.

Might want to count maps

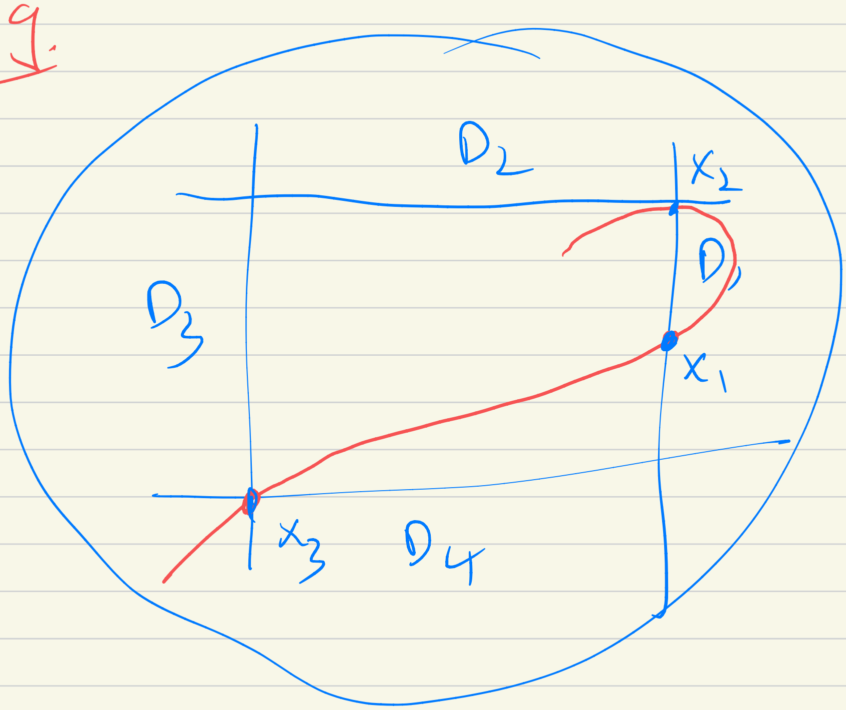
$$f : (C, x_1, \dots, x_n) \longrightarrow X$$

$x_1, \dots, x_n \in C$ distinct points

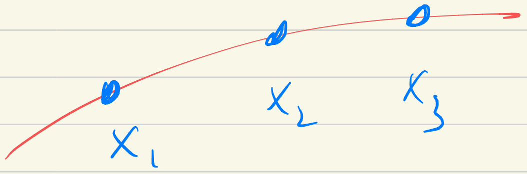
where we specify the contact order of f at a point x_i with each divisor D_j, with $\beta \in B(\mathbb{Z})$,

$\beta = \sum a_i D_i$, $a_i \geq 0$. specifying we should have contact order a_j with D_j.

e.g.



(C, x_1, x_2, x_3)



Contact orders at:

x_1 : D_1

x_2 : $D_1 + 2D_2$

x_3 : $D_3 + D_4$

Note: This produces counting problems where we count curves realizing such given tangency conditions.

(Log Gromov-Witten theory, Abramovich-Chen, Gr.-Siebert, 2010)

Assume: $\dim_{\mathbb{R}} B = \dim_{\mathbb{C}} X$, i.e.,

D has a 0-dim'l stratum ($\exists\, I$ s.t. $\dim D_I = 0$.)

Interesting case ($\mathbb{P}^2$, smooth elliptic curve)

Goal: We will construct a commutative ring $R(X, D)$ such that the mirror to $X \setminus D$ is $\operatorname{Spec} R(X, D)$

Actually, will construct a mirror family, and first need the base of this family.

Fix $P \subseteq H_2(X,\mathbb{Z})$ a submonoid

such that

- P contains the class of every holomorphic curve in X.
- $P \cap (-P) = H_2(X,\mathbb{Z})_{tors}$
- P saturated, i.e., $np \in P \Rightarrow p \in P$, for $p \in H_2(X,\mathbb{Z})$, $n \in \mathbb{Z}_{>0}$.
- P finitely generated.

Def:

$$A := \mathbb{C}[P] = \bigoplus_{p \in P} \mathbb{C} \cdot t^p$$

the monoid ring, with $t^p \cdot t^{p'} = t^{p+p'}$.

Fix a monomial ideal $I \subseteq A$ such that $A_I := A/I$ is Artinian

(i.e., finite dim'l $\mathbb{C}$-vector space)

Actually! Will construct, for each such I, a flat A_I-algebra $R_I(X,D)$, which will give a flat family

$$\begin{tikzcd} \operatorname{Spec} R_I(X,D) = \check{X}_I \arrow[d] \\ \operatorname{Spec} A_I \end{tikzcd}$$

Taking limit over all I

$$\begin{tikzcd} \check{\mathcal{X}} \arrow[d] & \arrow[l] \text{formal scheme} \\ \operatorname{Spf} \hat{A} & \arrow[l] \text{inverse limit over all } I. \end{tikzcd}$$

Construction of $R_I(X,D)$:

`\vartheta`

$$R_I(X,D) := \bigoplus_{p \in B(\mathbb{Z})} A_I \cdot \vartheta_p$$

Theta functions

This is the free A_I-module with generating set $B(\mathbb{Z})$.

Need to define an algebra structure.

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr} \vartheta_r$$

with $\alpha_{pqr} \in A_I = \mathbb{C}[P]/I$

$$\alpha_{pqr} = \sum_{A \in P \setminus I} N^A_{pqr} \cdot t^A$$

with $N^A_{pqr} \in \mathbb{Q} \subseteq \mathbb{C}$

Key point is the def'n of N^A_{pqr}.

Def: For $r \in B(\mathbb{Z})$, $r = \sum_{i \in I} a_i D_i$, $a_i > 0$

gives a stratum D_I of D,

which we write as D_r.

Fix a point $z \in D_r$. We define

N^A_{pqr} = # of maps

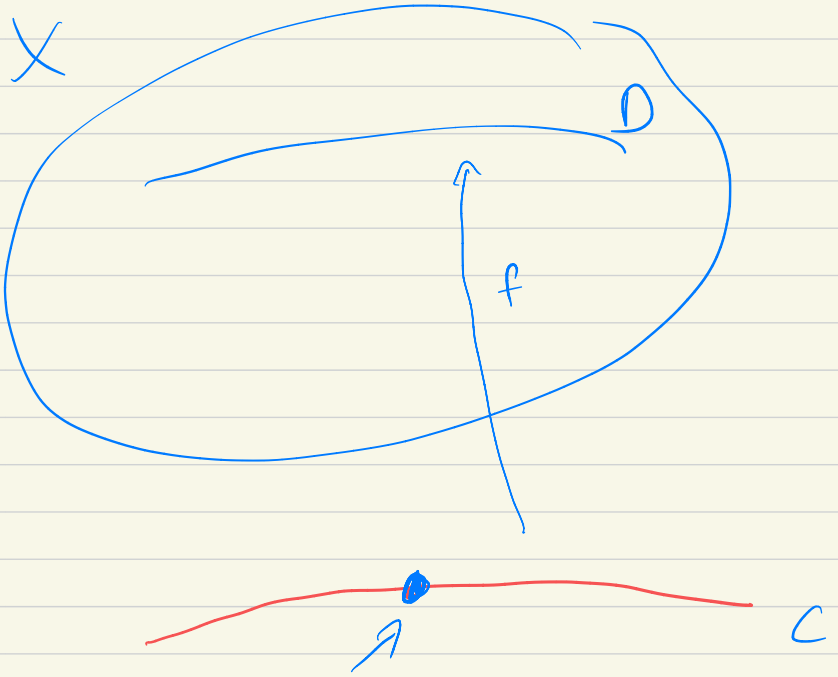
$$f : (C, x_1, x_2, x_{out}) \longrightarrow X$$

with

- C genus 0
- $f_* [C] = A$
- At x_1, f has contact order given by p
- At x_2, f has contact order given by q
- At x_{out}, $f(x_{out}) = z$ and f has contact order given by $-r$.

This involves a notion of *negative contact order*. To define properly, we use *punctured log Gromov-Witten theory* (Abramovich, Chen, G., Siebert, 2020.)

e.g. X a surface, $D \subseteq X$ smooth rational curve with $D^2=-1$



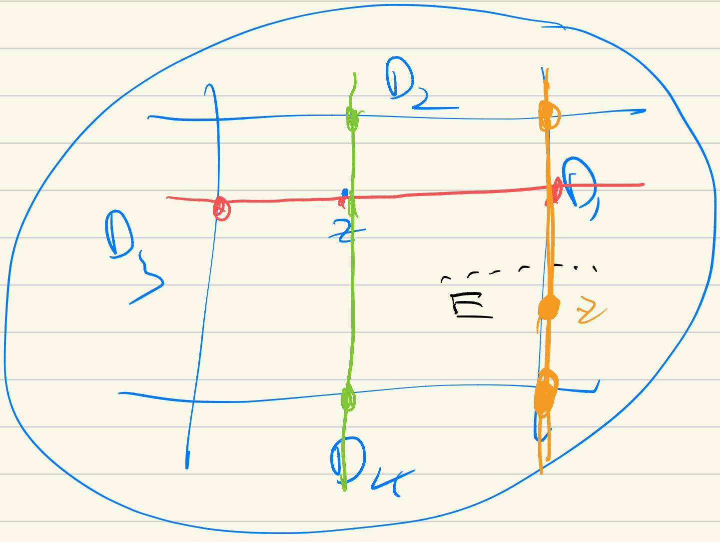
$f_*[C]=D$

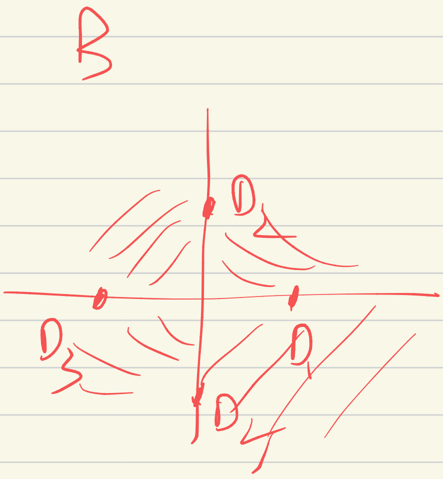
This point can be given contact order -1, so all negative intersection number is concentrated at this one point.

Theorem (G-S, '19) The numbers N^{β}_{pqr} can be defined rigorously, and give $R_I(X,D)$ the structure of an associative, commutative A_I-algebra with unit $1 = \vartheta_0$.

Example:

Blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$





Two key calculations: $\vartheta_{D_1}\cdot\vartheta_{D_3}$

and $\vartheta_{D_2}\cdot\vartheta_{D_4}$.

$$\boxed{\begin{aligned}\vartheta_{D_1}\cdot\vartheta_{D_3} &= N^{D_2}_{D_1 D_3 0}\, t^{D_2}\cdot\vartheta_0 \\ &= t^{D_2}\end{aligned}}$$

$D_0 = X$

$$\vartheta_{D_2}\cdot\vartheta_{D_4} = t^{D_2}\cdot\vartheta_0 + \underbrace{N^{D_1}_{D_2 D_4 D_1}}_{=1}\cdot t^{D_1}\cdot\vartheta_{D_1}$$

$$\boxed{\vartheta_{D_2}\cdot\vartheta_{D_4} = t^{D_2} + t^{D_1}\vartheta_{D_1}}$$

These 2 equations determine the <u>mirror family</u>!

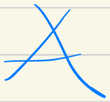
Remark: In certain case, ∃ algorithms for carrying these calculations out, using the technology of scattering diagrams [Argüz-G, 2020] tells us the mirrors whenever (X, D) is obtained by blowing up a toric variety.

(Builds on G-Siebert '21 "The canonical wall structure.")

$(\mathbb{P}^1, D_1+D_2)$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[d] & (x,y) \arrow[d, mapsto] \\ \mathbb{C} & x\cdot y \end{tikzcd}$$

Givental mirror.

$(\mathbb{P}^2,$  $)$

$$\begin{tikzcd} \mathbb{C}^3 \arrow[d] & (x,y,z) \arrow[d, mapsto] \\ \mathbb{C} & xyz \end{tikzcd}$$

$\operatorname{Spec} \dfrac{\mathbb{C}[t][\theta_1,\theta_2,\theta_3]}{(\theta_1\theta_2\theta_3 - t)}$