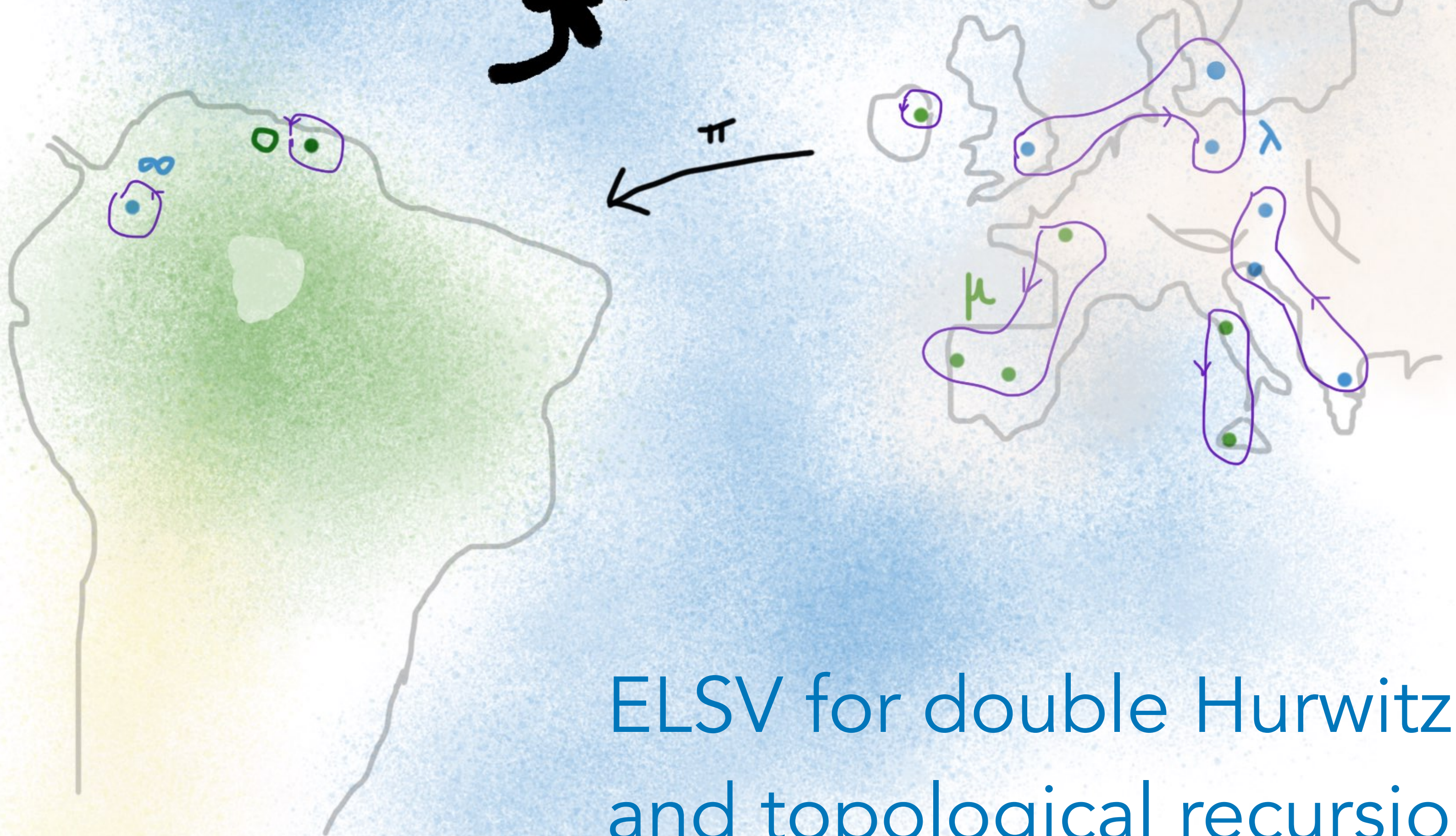
Algebraic geometry seminar
Bogota

$$\int_{\overline{\mathcal{M}}_{g,n}} \Omega \prod_{i=1}^{n} \frac{1}{1-\mu_i\psi_i}$$



ELSV for double Hurwitz numbers and topological recursion

Gaëtan Borot
HU Berlin

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I. Introduction to Hurwitz numbers

- representation theory of the symmetric group
- combinatorial approach

II. Intersection theory and topological recursion

- simple Hurwitz numbers
- orbifold Hurwitz numbers
- double Hurwitz numbers

I

Introduction to Hurwitz numbers

I. Introduction to Hurwitz numbers — Definition

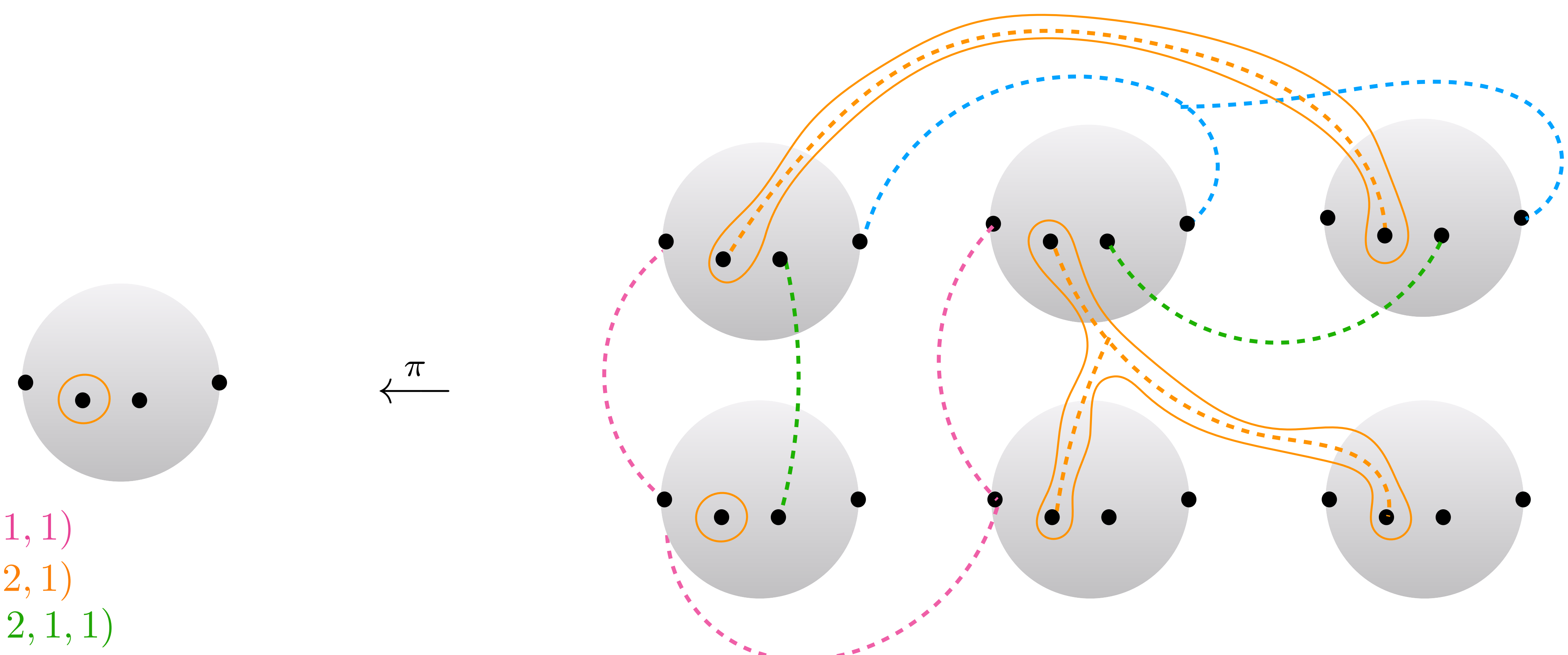
Topological branched covers $\Sigma \xrightarrow{L:1} \mathbb{S}^2$ with branchpoints $y_1, \dots, y_{k+2} \in \mathbb{S}^2$ modulo automorphisms

$$\Updownarrow$$

Iso. class of representations $\rho \, : \, \pi_1(\mathbb{S}^2 \setminus Y) \cong \big\langle (\gamma_i)_{i=1}^{k+2} \mid \gamma_1 \cdots \gamma_{k+2} = 1 \big\rangle \longrightarrow \mathfrak{S}_L$

Riemann-Hurwitz formula $\chi(\Sigma) = 2L - \sum_{j=1}^{k+2} (L - \ell(\lambda_j))$ where $C_{\lambda_j} = [\rho(\lambda_j)]$

$\chi = 2$



$\lambda_1 = (4, 1, 1)$
$\lambda_2 = (3, 2, 1)$
$\lambda_3 = (2, 2, 1, 1)$
$\lambda_4 = (3, 1, 1, 1)$

Topological branched covers $\Sigma \xrightarrow{L:1} \mathbb{S}^2$ with branchpoints $y_1, \ldots, y_{k+2} \in \mathbb{S}^2$ modulo automorphisms

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Riemann-Hurwitz formula $\quad \chi(\Sigma) = 2L - \sum_{j=1}^{k+2} (L - \ell(\lambda_j))$ where $C_{\lambda_j} = [\rho(\lambda_j)]$

- The partition $\lambda_j \vdash L$ is the ramification profile above y_j
- If $\rho(\gamma_j)$ is a transposition, one says that y_j is a simple branchpoint
- One often considers the first (resp. last) branchpoint to be 0 (resp. ∞)

$Z(\mathbb{Q}[\mathfrak{S}_L])$ = center of the symmetric group algebra

$\mathcal{B} := \lim_{\infty \leftarrow N} \mathbb{Q}[x_1, \ldots, x_N]^{\mathfrak{S}_N}$ = ring of symmetric functions in countably many variables

$$\cong \bigoplus_{L \geq 0} Z(\mathbb{Q}[\mathfrak{S}_L])$$

Two bases, indexed by $\lambda \vdash L$

Conjugacy classes

$$\hat{C}_\lambda = \sum_{\gamma \in C_\lambda} \gamma$$

$$\hat{C}_{\lambda_1} \hat{C}_{\lambda_2} = \sum_{\lambda_3 \vdash L} \left|\{\gamma_1\gamma_2\gamma_3 = \mathrm{id} \mid \gamma_i \in C_{\lambda_i}\}\right| \frac{C_{\lambda_3}}{|C_{\lambda_3}|}$$

power sums $\quad \frac{p_\lambda}{L!} \longleftrightarrow \frac{C_\lambda}{|C_\lambda|}$

Idempotents

$$\hat{\Pi}_\lambda = \frac{\chi_\lambda(\mathrm{id})}{L!} \sum_{\mu \vdash L} \chi_\lambda(C_\mu)\, \hat{C}_\mu$$

$$\hat{\Pi}_\lambda \hat{\Pi}_\mu = \delta_{\lambda,\mu} \hat{\Pi}_\lambda$$

Schur polynomials $\quad \frac{s_\lambda}{L!} \longleftrightarrow \frac{\hat{\Pi}_\lambda}{\chi_\lambda(\mathrm{id})}$

- Jucys-Murphy elements $\hat{J}_k = \sum_{i=2}^{k} (i\,k)$

  Symmetric polynomials evaluated on $(\hat{J}_k)_{k=2}^{L}$ span $Z(\mathbb{Q}(\mathfrak{S}_L))$

- If r is a symmetric polynomial, one defines Hurwitz numbers in two versions

  $$R^{\bullet}_{\mu,\nu} = \frac{1}{L!}\,[\mathrm{id}]\,\hat{C}_\mu\, r(\hat{J},0,0,\ldots)\,\hat{C}_\nu \quad \rightsquigarrow \quad R^{\circ}_{\mu,\nu}$$ by inclusion-exclusion

$\boxed{r = e^{\beta p_1}}$ **Double Hurwitz numbers**

$$R^{\bullet}_{\mu,\nu} = \frac{1}{L!}\sum_{k \geq 0} \frac{\beta^k}{k!} \sum_{\substack{\tau_1,\ldots,\tau_k \\ \text{transpositions}}} [\mathrm{id}]\,\hat{C}_\mu \tau_1 \cdots \tau_k \hat{C}_\nu = \frac{1}{L!}\sum_{k\geq 0}\left|\left\{ \begin{array}{l} \gamma_0 \in C_\mu \\ \gamma_\infty \in C_\nu \end{array} \quad \tau_i \text{ transposition} \;\middle|\; \gamma_0\tau_1\cdots\tau_k\gamma_\infty = \mathrm{id}\right\}\right| \cdot \frac{\beta^k}{k!}$$

is the weighted count of branched covers with ramification profile μ above 0 , ν above ∞ , k simple branchpoints

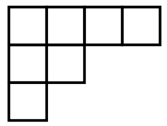
$R^{\circ}_{\mu,\nu}$ counts only the connected covers

k determines the genus by $2 - 2g = \ell(\mu) + \ell(\nu) - k$

- Symmetric polynomials in Jucys Murphy act diagonally on the idempotent basis

$$r(\hat{J},0,0,\dots)\hat{\Pi}_\nu = r(\mathrm{cont}\,\nu,0,0,\dots)\hat{\Pi}_\nu$$

$$\mathrm{cont}\,\nu = \{j-i \mid (i,j) = \text{box in } \nu\}$$

$\nu =$ 

$$\mathrm{cont}\,\nu = (0,1,2,3,-1,0,-2)$$

- Cauchy identity

$$\sum_\nu s_\nu \otimes s_\nu = \sum_\nu \frac{|C_\nu|}{|\nu|!}\, p_\nu \otimes p_\nu$$

$r(\hat{J},0,0,\dots)\otimes \mathrm{id}$

$$\sum_\nu r(\mathrm{cont}\,\nu)\, s_\nu \otimes s_\nu = \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} R^\bullet_{\mu,\nu} p_\mu \otimes p_\nu$$

e.g. $p_1(\mathrm{cont}\,\nu,0,0,\dots) = \sum_{i=1}^{\ell(\nu)}\sum_{j=1}^{\nu_i}(j-i) = \frac{1}{2}\sum_{i=1}^{\ell(\nu)}\left[\left(\nu_i - i + \tfrac{1}{2}\right)^2 - \left(-i+\tfrac{1}{2}\right)^2\right] := c_2(\nu)$

$\Longrightarrow$ double Hurwitz numbers are encoded in the decomposition of

$Z := \sum_\nu e^{\beta c_2(\nu)}\, s_\nu \otimes s_\nu$ on the power sum basis

for connected : $\ln Z$

Problem 1 : compute connected Hurwitz numbers for fixed topology

i.e. fixed genus g and $n = \ell(\nu)$ points above ∞

- Simple Hurwitz numbers : 0 unramified, i.e. $\mu = (1, \ldots, 1)$

$$H_{g,n}(\nu_1, \ldots, \nu_n) := \left[\beta^{2g-2+n+\sum_i \nu_i}\right] R^\circ_{(1,\ldots,1),\nu}$$

- d-orbifold Hurwitz numbers : $\mu = (d, \ldots, d)$ above 0

$$H^{[d]}_{g,n}(\nu_1, \ldots, \nu_n) := \left[\beta^{2g-2+n+\sum_i \frac{\nu_i}{d}}\right] R^\circ_{(d,\ldots,d),\nu}$$

- double Hurwitz numbers : arbitrary ramification above 0 and ∞

$$\mathbb{H}_{g,n}(\nu_1, \ldots, \nu_n) = \sum_{\mu \vdash |\nu|} \left[\beta^{2g-2+n+\ell(\mu)}\right] R^\circ_{\mu,\nu}\, \vec{q}_\mu \qquad \text{where} \quad \vec{q}_\mu = \prod_{i=1}^{\ell(\mu)} q_{\mu_i}$$

For simple Hurwitz numbers (0 unramified), one can try elementary combinatorics

Covers are in (orbifold) bijection with their **branching graph**

This is a ribbon graph with labeled edges

(Okounkov, Pandharipande, 01)

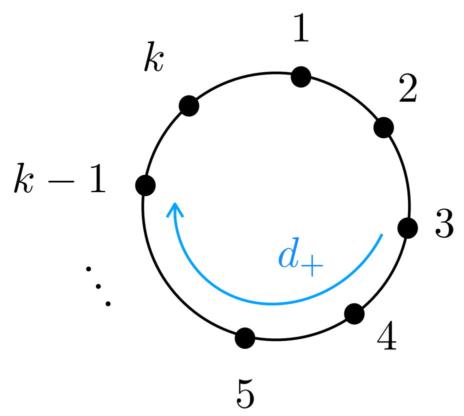
vertices $\longleftrightarrow$ sheets

edges labeled by $[\![1, k]\!]$ $\longleftrightarrow$ simple branchpoints

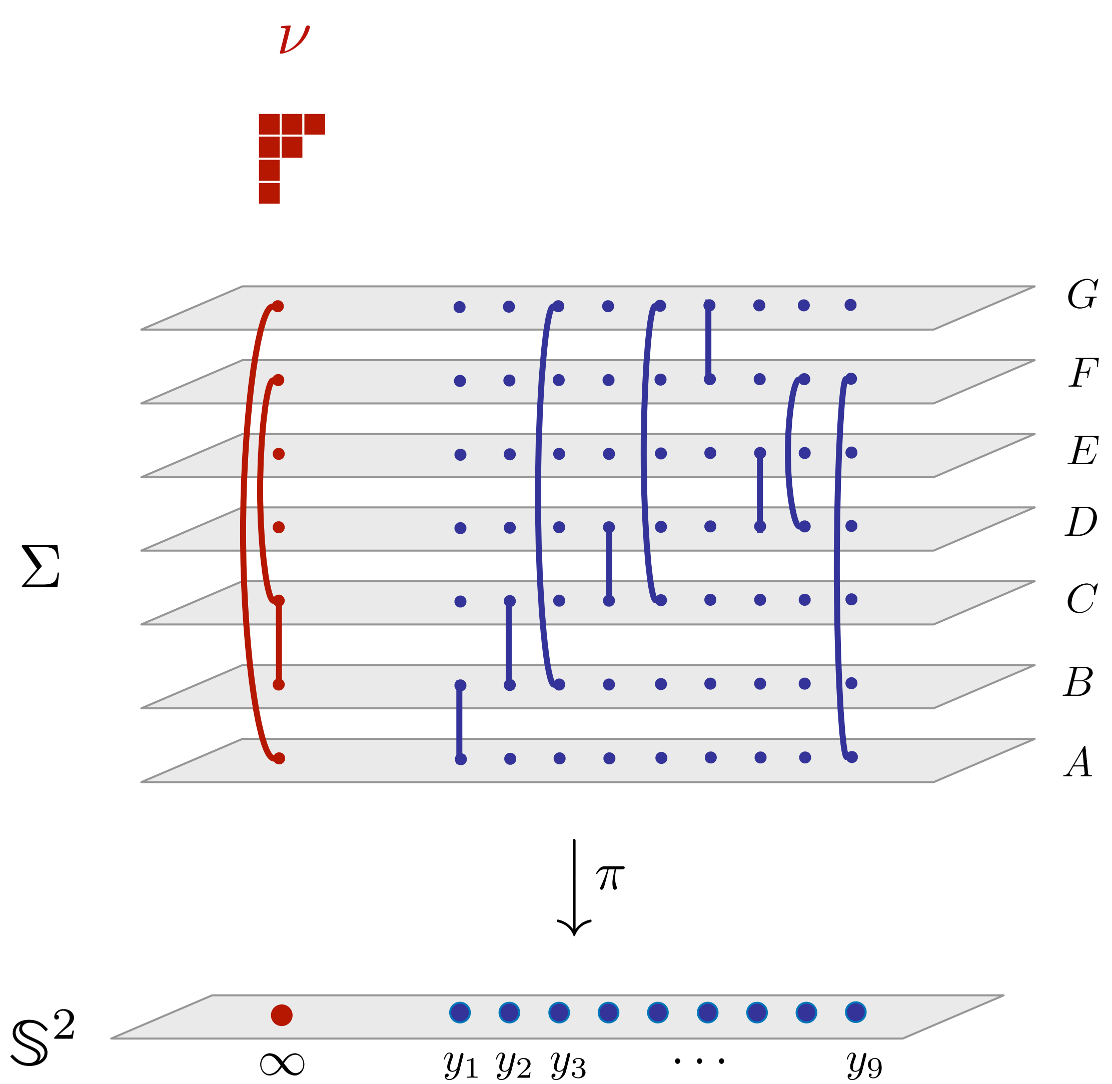
faces $\longleftrightarrow$ points above ∞

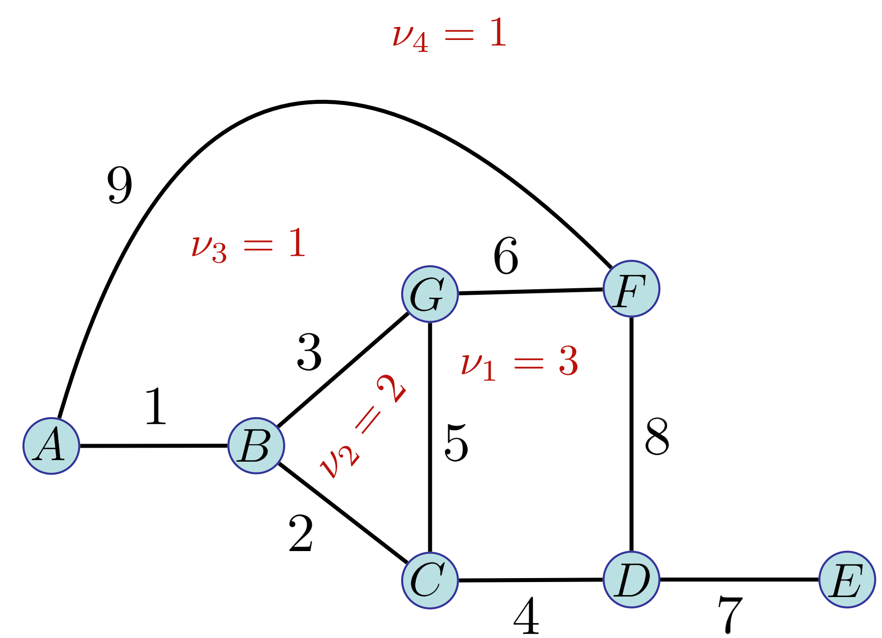
$\frac{1}{k} \sum_{\substack{e \text{ around} \\ \text{face } f}} d_+(l_e, l_{e'})$ $\longleftrightarrow$ multiplicity of point f

$\rightsquigarrow \nu$



Example





$g = 0$

g = 0, n = 1 face $\nu = (\nu_1)$

Branched trees with ν_1 vertices $\rightsquigarrow$ $H_{0,1}(\nu_1) = \dfrac{\nu_1^{\nu_1 - 2}}{\nu_1!}$ (Cayley 1899)

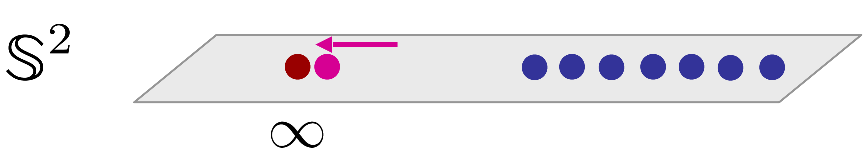
g = 0, n = 2 faces

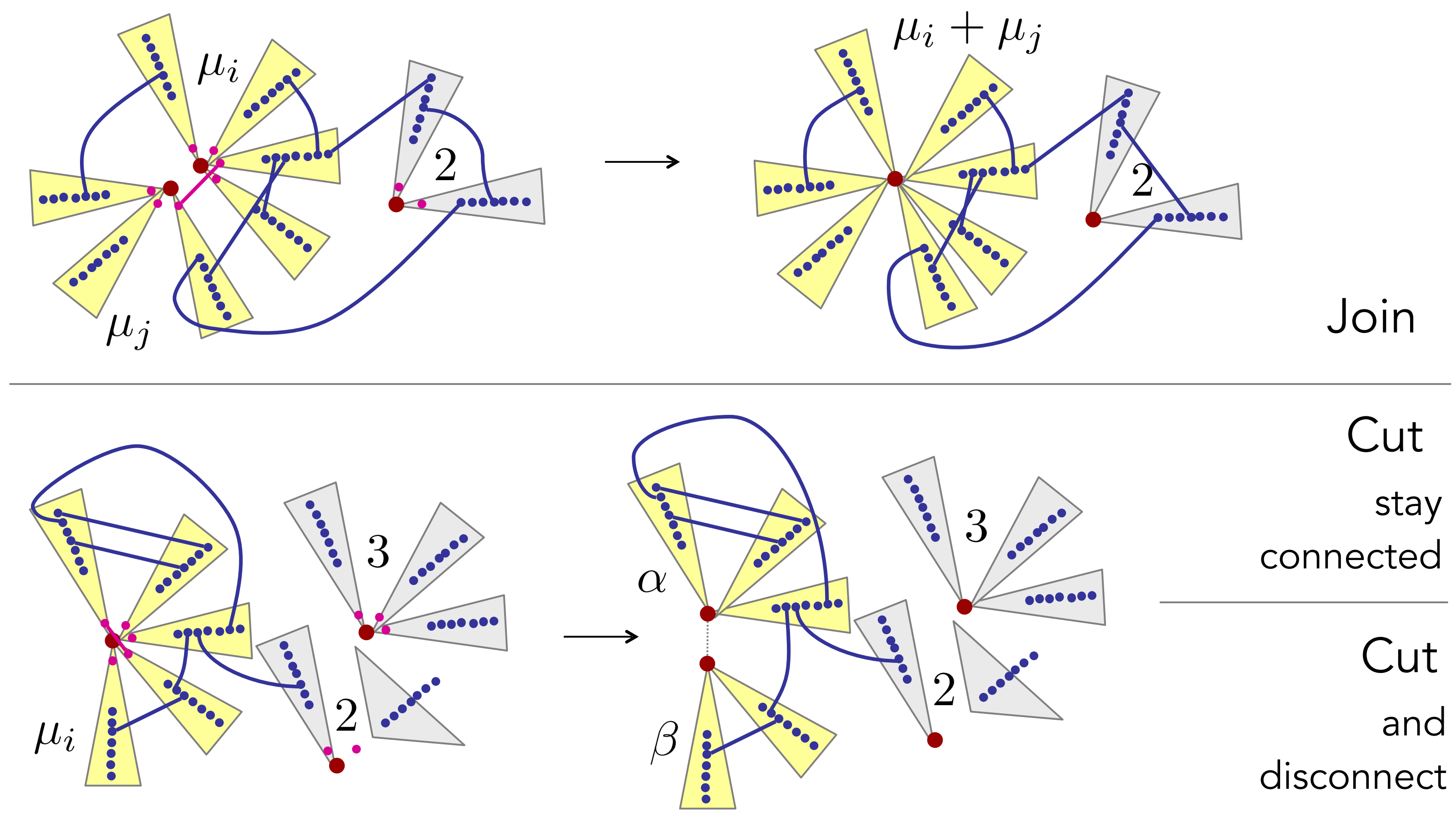
Branched graph with 1 cycle + trees $\rightsquigarrow$ $H_{0,2}(\nu_1, \nu_2) = \dfrac{\nu_1^{\nu_1} \nu_2^{\nu_2}}{\nu_1! \nu_2! (\nu_1 + \nu_2)}$

I. Introduction to Hurwitz numbers — Combinatorial approach

Cut-and-join equation (Goulden, Jackson, Vakil, 99)

Recursion on the number of edges by merging a simple branchpoint to ∞





$$(2g-2+\ell(\nu)+|\nu|)\,H_{g,n}(\nu)$$

$$= \sum_{i<j} (\nu_i+\nu_j)\,\frac{(\nu_i+\nu_j)!}{\nu_i!\nu_j!}\,H_{g,n-1}\big(\nu\setminus\{\nu_i,\nu_j\}\sqcup(\nu_i+\nu_j)\big)$$

$$+\frac{1}{2}\sum_{i=1}^{\ell(\nu)}\sum_{m+m'=\nu_i} m\,m'\,\frac{m!m'!}{\nu_i!}\Bigg(H_{g-1,n+1}(\nu\setminus\{\nu_i\}\sqcup m, m') + \sum_{\substack{h+h'=g\\ \lambda\sqcup\lambda'=\nu\setminus\{\nu_i\}}} H_{h,\ell(\lambda)+1}(\lambda\sqcup m)H_{g_b,\ell(\lambda')+1}(\lambda'\sqcup m')\Bigg)$$

I. Introduction to Hurwitz numbers — Combinatorial approach

The cut-and-join equation determines all the $H_{g,n}(\nu)$

Problem 1' : can one compute all of them for fixed (g,n) at the same time ?

$$W_{g,n}(x_1,\ldots,x_n) = \sum_{\nu_1,\ldots,\nu_n\geq 1} H_{g,n}(\nu_1,\ldots,\nu_n)\prod_{i=1}^{n}\mathrm{d}(e^{\mu_i x_i})$$

We already have found

$$W_{0,1}(x_1) = \sum_{\nu_1\geq 1}\frac{\nu_1^{\nu_1-2}}{\nu_1!}\,\mathrm{d}(e^{\nu_1 x_1}) = y_1\,\mathrm{d}x_1 \qquad \text{with} \qquad e^{x_i} = y_i e^{-y_i}$$

$$W_{0,2}(x_1,x_2) = \sum_{\nu_1,\nu_2\geq 1}\frac{\nu_1^{\nu_1}\nu_2^{\nu_2}\,\mathrm{d}(e^{\nu_1 x_1})\mathrm{d}(e^{\nu_2 x_2})}{\nu_1!\nu_2!(\nu_1+\nu_2)} = \frac{\mathrm{d}y_1\mathrm{d}y_2}{(y_1-y_2)^2} - \frac{\mathrm{d}(e^{x_1})\mathrm{d}(e^{x_2})}{(e^{x_1}-e^{x_2})^2}$$

Observe the role of the Lambert curve $\mathcal{S}\,:\,e^x = ye^{-y}$

This combinatorial approach can be adapted to other Hurwitz problems but
- there are more powerful algebraic techniques doing the same
- cut-and-join does not easily give access to $W_{g,n}$ for $2g-2+n>0$

II

Intersection theory and topological recursion

Simple Hurwitz numbers from the viewpoint of algebraic geometry

$\mathcal{H}_{g,\nu_1,\ldots,\nu_n}$ = moduli space of genus g Riemann surfaces $(C, p_1, \ldots, p_n, z_1, \ldots, z_k)$
equipped with $\pi : C \to \mathbb{P}^1$ having poles of order ν_i at p_i
such that $\mathrm{d}\pi$ has simple zeroes at z_j

where $k = 2g - 2 + n + \sum_i \nu_i$

$$\mathcal{X} : \begin{array}{ccc} \mathcal{H}_g(\nu_1, \ldots, \nu_n) & \longrightarrow & \mathbb{C}^k/\mathbb{C} \\ \big[(C, \mathbf{p}, \mathbf{z}, \pi)\big] & \longmapsto & [\pi(\mathbf{z})] \end{array}$$

(modulo translation)

$$\Longrightarrow \quad H_{g,n}(\nu_1, \ldots, \nu_n) = \frac{\deg \mathcal{X}}{k!}$$

After dealing with compactification, this can be computed as the total Segre class of a cone $\overline{\mathcal{H}}_{g,\nu_1,\ldots,\nu_n} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$\overline{\mathcal{M}}_{g,n}$ = Deligne-Mumford compactification of the moduli space of curves

II. Intersection theory and topological recursion — Simple Hurwitz numbers

$\overline{\mathcal{M}}_{g,n}$ = Deligne-Mumford compactification of the moduli space of curves

Cotangent line bundle

$$(\mathbb{L}_i)_{(C,p_1,\ldots,p_n)} = T^*_{p_i}C \quad \rightsquigarrow \quad \psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

Hodge bundle

$$E_{(C,p_1,\ldots,p_n)} = H^0(C,\omega_C) \quad \overset{\text{total Chern class}}{\rightsquigarrow} \quad \Lambda^\vee = c(\mathbb{E}^\vee) = \sum_{h=0}^{g} (-1)^h \lambda_h \in H^*(\mathcal{M}_{g,n})$$

Theorem (Ekedahl, Lando, Shapiro, Vainshtein, 01)

For $2g-2+n>0$

$$H_{g,n}(\nu_1,\ldots,\nu_n) = \prod_{i=1}^{n} \frac{\nu_i^{\nu_i}}{\nu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda^\vee}{\prod_{i=1}^{n}(1-\nu_i\psi_i)}$$

combinatorial prefactor

polynomial in $\nu_1,\ldots,\nu_n$

Gave some explicit evaluations of Hodge integrals …

II. Intersection theory and topological recursion — Simple Hurwitz numbers

Topological recursion allows computing the generating series $W_{g,n}$
from periods on the Lambert curve $\mathcal{S} \, : \, e^x = ye^{-y}$

Conjecture : Bouchard, Mariño conjecture 08

Theorem : Eynard, Mulase, Safnuk 11

coming from mirror symmetry

A

Gromov-Witten invariants of
$\mathbb{C}^3$ in the infinite framing limit

B

Period computations on the mirror
$\mathcal{S} \, : \, e^x = ye^{-y}$

Consequence of a more general correspondence for all toric CY3 folds

Conjecture : Bouchard, Klemm, Mariño, Pasquetti 07

Theorem : Eynard, Orantin 15 (also Fang, Liu, Zong, 16)

Topological recursion (TR)

- Initial data consists of

$$\begin{cases} \mathcal{S} & \text{smooth complex curve} \\ x, y & \text{meromorphic functions} \\ B \in H^0\big(\mathcal{S}^2, K_{\mathcal{S}}^{\boxtimes 2}(2\Delta)\big)^{\mathfrak{S}_2} \end{cases}$$

assuming

$\mathrm{d}x$ has simple zeros $\mathfrak{a}$

y has order 1 at $\mathfrak{a}$

- σ_a local involution near $a \in \mathfrak{a}$ such that $x \circ \sigma_a = x$

- Recursion kernel $K(z_1, z) = \dfrac{1}{2} \dfrac{\int_{\sigma_a(z)}^{z} \omega_{0,2}(\cdot, z_1)}{\big(y(z) - y(\sigma_a(z))\big)\mathrm{d}x(z)}$

- TR then constructs $\omega_{0,1} = y\,\mathrm{d}x$, $\omega_{0,2} = B$ and $\omega_{g,n} \in H^0\big(\mathcal{S}^n, K_{\mathcal{S}}^{\boxtimes n}(*\mathfrak{a})\big)^{\mathfrak{S}_n}$

  by induction on $2g - 2 + n > 0$

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{a \in \mathfrak{a}} \operatorname*{Res}_{z=a} K(z_1, z) \Bigg(\omega_{g-1,n+1}(z, \sigma_a(z), z_2, \ldots, z_n) + \sum_{\substack{h+h'=g \\ J \sqcup J' = \{z_2, \ldots, z_n\}}}^{\text{no } \omega_{0,1}} \omega_{h,1+|J|}(z, J)\, \omega_{h',1+|J'|}(\sigma_a(z), J') \Bigg)$$

Topological recursion (TR)

construct $\omega_{0,1} = y\,\mathrm{d}x$, $\omega_{0,2} = B$ and $\omega_{g,n} \in H^0\big(\mathcal{S}^n, K_{\mathcal{S}}^{\boxtimes n}(*\mathfrak{a})\big)^{\mathfrak{S}_n}$

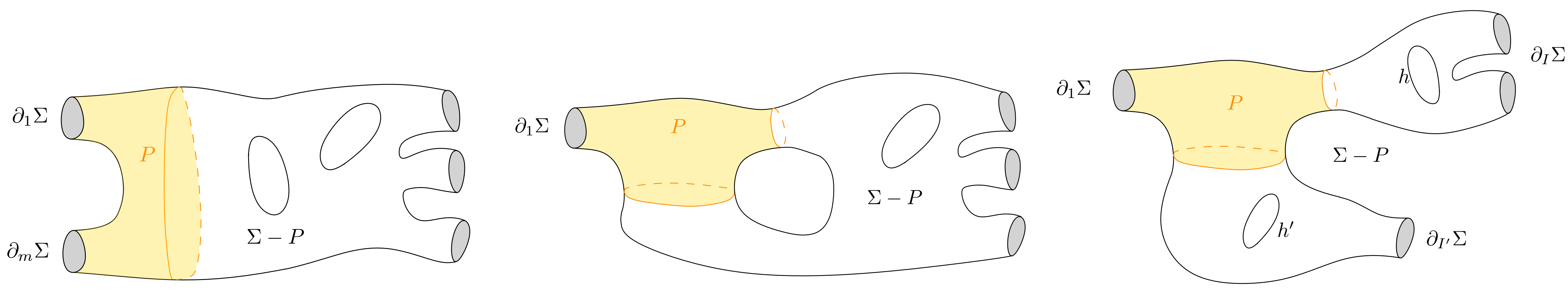
by induction on $2g-2+n>0$, computing residues

The terms in TR are in bijection with

$$\overline{\mathcal{P}}_{\Sigma} = \Big\{ P \hookrightarrow \Sigma \text{ such that } \partial_1 P = \partial_1 \Sigma \text{ and } \Sigma - P \text{ stable} \Big\} \Big/ \mathrm{Diff}_{\Sigma}^{\partial}$$

Σ = smooth surface of genus g with n labeled boundaries

P = pair of pants



II. Intersection theory and topological recursion — Simple Hurwitz numbers

Theorem (Eynard, Mulase, Safnuk, 11)

Let $\omega_{g,n}$ be the outcome of TR for

$$\mathcal{S} = \{(x,y) \in \mathbb{C} \times \mathbb{C}^* \quad | \quad e^x = ye^{-y}\} \text{ with } B = \frac{\mathrm{d}y_1 \mathrm{d}y_2}{(y_1 - y_2)^2}$$

We have the identity of series expansion near $e^{x_i} \to 0$

$$\omega_{g,n} - \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}(e^{x_1})\mathrm{d}(e^{x_2})}{(e^{x_1} - e^{x_2})^2} \sim W_{g,n}(x_1, \ldots, x_n)$$

$$\mathfrak{a} = \{(-1,1)\}$$

- Proof by analysis of cut-and-join relations + analytic continuation to $\mathcal{S}$

- The outcome of TR can be always be represented via intersection theory on $\overline{\mathcal{M}}_{g,n}$ Eynard 11

  $\rightsquigarrow$ algebraic/combinatorial proof of the ELSV formula

  (Hodge class appears by Mumford formula)

d -orbifold Hurwitz numbers : $\mu = (d, \ldots, d)$ above 0

$$H_{g,n}^{[d]}(\nu_1, \ldots, \nu_n) := \left[\beta^{2g-2+n+\sum_i \frac{\nu_i}{d}}\right] R^{\circ}_{(d,\ldots,d),\nu}$$

$$W_{g,n}^{[d]}(x_1, \ldots, x_n) = \sum_{\nu_1,\ldots,\nu_n \geq 1} H_{g,n}^{[d]}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \mathrm{d}(e^{\nu_i x_i})$$

There is a cut-and-join equation, from which one can compute

$$W_{0,1}^{[d]}(x_1) = y_1 \, \mathrm{d}x_1$$

$$W_{0,2}^{[d]}(x_1, x_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} - \frac{\mathrm{d}(e^{x_1})\mathrm{d}(e^{x_2})}{\left(e^{x_1} - e^{x_2}\right)^2}$$

$$\mathcal{S}^{[d]} \ : \left\{ \begin{array}{lcl} x(z) & = & \ln z - z^d \\ y(z) & = & z^d \end{array} \right.$$

From the algebraic geometry viewpoint, it can be approached via intersection theory on $\overline{\mathcal{M}}_{g,a_1,\ldots,a_n}(B\mathbb{Z}_d)$ $a_i \in \mathbb{Z}_d$ monodromy at p_i

$\mathbb{E}^U \longrightarrow \overline{\mathcal{M}}_{g,a_1,\ldots,a_n}$ subbundle of the Hodge bundle on which the generator of $\mathbb{Z}_d$ acts by $e^{2\mathrm{i}\pi/d}$

$$\Lambda^{U\vee} = c(\mathbb{E}^{U\vee}) = \sum_{h=0}^{g}(-1)^k \lambda_k^U \in H^*\big(\mathcal{M}_{g,a_1,\ldots,a_n}(B\mathbb{Z}_d)\big)$$

Theorem (Johnson, Pandharipande, Tseng 11)

For $2g-2+n>0$

$$H_{g,n}^{[d]}(\nu_1,\ldots,\nu_n) = d^{2g-2+n+\sum_i \frac{\nu_i}{d}} \prod_{i=1}^{n} \frac{\left(\frac{\nu_i}{d}\right)^{\lfloor \frac{\nu_i}{d} \rfloor}}{\lfloor \frac{\nu_i}{d} \rfloor !} \int_{\overline{\mathcal{M}}_{g,-\overline{\nu}_1,\ldots,-\overline{\nu}_n}(B\mathbb{Z}_d)} \frac{\Lambda^{U\vee}}{\prod_{i=1}^{n}\left(1-\frac{\nu_i}{d}\psi_i\right)}$$

$-\overline{\nu_i} \in [\![0,d-1]\!]$ such that $-\overline{\nu_i} = -\nu_i \mod d$

combinatorial prefactor

quasi-polynomial in $\nu_1,\ldots,\nu_n$

There is an equivalent description via the moduli stacks of $\frac{s}{r}$ th roots

$$\overline{\mathcal{M}}_{g,n}^{(r,s)}(a_1,\ldots,a_n) = \overline{\left\{(C,p_1,\ldots,p_n,L,\phi) \;\middle|\; L^{\otimes r} \overset{\phi}{\simeq} \omega_C^{\otimes s}\Big(\sum_i (s-a_i)p_i\Big)\right\}} \Big/ \sim$$

$$\mathcal{L} \longrightarrow \mathcal{C} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}^{(r,s)}(a_1,\ldots,a_n) \xrightarrow{\epsilon} \overline{\mathcal{M}}_{g,n}$$

universal line bundle — universal curve

↳ Chiodo classes

$$\Omega_{g,n}^{(r,s)}(a_1,\ldots,a_n) := \epsilon_* c(-R^*\pi_*\mathcal{L}) \in H^*(\overline{\mathcal{M}}_{g,n})$$

Theorem (Dunin-Barkowski, Lewanski, Popolitov, Shadrin 15)

$$H_{g,n}^{[d]}(\nu_1,\ldots,\nu_n) = d^{2g-2+n+\sum_i \frac{\nu_i}{d}} \prod_{i=1}^{n} \frac{\left(\frac{\nu_i}{d}\right)^{\lfloor \frac{\nu_i}{d} \rfloor}}{\lfloor \frac{\nu_i}{d} \rfloor !} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega_{g,n}^{(d,d)}(-\overline{\nu}_1,\ldots,-\overline{\nu}_n)}{\prod_{i=1}^n \left(1 - \frac{\nu_i}{d}\psi_i\right)}$$

$-\overline{\nu_i} \in [\![0, d-1]\!]$ such that $-\overline{\nu_i} = -\nu_i \mod d$

combinatorial prefactor

quasi-polynomial in $\nu_1,\ldots,\nu_n$

Theorem (Do, Leigh, Norbury, 12 | Bouchard, Hernandez-Serrano, Liu, Mulase 13)

Let $\omega_{g,n}^{[d]}$ be the outcome of TR for

$$\mathcal{S}^{[d]} : \begin{cases} x(z) & = & \ln z - z^d \\ y(z) & = & z^d \end{cases} \quad \text{with} \quad B = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} \qquad \mathfrak{a} = \left\{ z = e^{2\mathrm{i}\pi/d} d^{1/d} \mid j \in \mathbb{Z}_d \right\}$$

We have the identity of series expansion near $e^{x_i} \to 0$

$$\omega_{g,n}^{[d]} - \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}(e^{x_1})\mathrm{d}(e^{x_2})}{\left(e^{x_1} - e^{x_2}\right)^2} \sim W_{g,n}^{[d]}(x_1, \ldots, x_n)$$

- Proof by analysis of cut-and-join relations + analytic continuation to
- The outcome of TR can be always be represented via intersection theory on $\overline{\mathcal{M}}_{g,n}$ Eynard 11

  ⇝ algebraic/combinatorial proof of the JPT formula

  Dunin-Barkowski, Lewanski, Popolitov, Shadrin 15

Double Hurwitz numbers : arbitrary ramification above 0 and ∞

$$\mathbb{H}_{g,n}(\nu_1,\ldots,\nu_n) = \sum_{\mu \vdash |\nu|} \left[\beta^{2g-2+n+\ell(\mu)}\right] R^{\circ}_{\mu,\nu}\,\vec{q}_\mu \qquad \text{where} \qquad \vec{q}_\mu = \prod_{i=1}^{\ell(\mu)} q_{\mu_i}$$

$$\mathbb{W}_{g,n}(x_1,\ldots,x_n) = \sum_{\nu_1,\ldots,\nu_n \geq 1} \mathbb{H}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \mathrm{d}(e^{\nu_i x_i})$$

- Double Hurwitz numbers are more fundamental than the others
  Relate to intersection theory of the double ramification cycle
  which has applications to integrability, structural properties of $H^*(\overline{\mathcal{M}}_{g,n})$

- There are vague conjectures about ELSV-like formulas involving
  unspecified moduli stacks of $\dim_{\mathbb{C}} = 4g-3+n$ with a proper fibration to $\overline{\mathcal{M}}_{g,n}$
  and unspecified analogs of the dual Hodge class
  Goulden, Jackson 03 ; Bayer, Cavalieri, Johnson, Markwig 11

- There is an (elementary) cut-and-join equation, from which one can find

$$\mathbb{W}_{0,1}(x_1) = y_1\,\mathrm{d}x_1$$

$$\mathbb{W}_{0,2}(x_1,x_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2} - \frac{\mathrm{d}(e^{x_1})\mathrm{d}(e^{x_2})}{\left(e^{x_1}-e^{x_2}\right)^2}$$

$$\mathcal{S}_{\mathbf{q}}\ :\ \begin{cases} x(z) & = & \ln z - \sum_j q_j z^j \\ y(z) & = & \sum_j q_j z^j \end{cases}$$

- When $q_1,\ldots,q_{d-1},q_d$ generic
  $q_d \neq 0$
  $q_j = 0$ for $j > d$

$\mathrm{d}x$ has $|\mathfrak{a}| = d$ simple zeros

which are the roots of $1 - \sum_{j=1}^{d} j q_j z^j = 0$

can be seen as a deformation of the d-orbifold case $q_j = \delta_{j,d}$

Theorem 1 (B., Do, Karev, Lewanski, Moskowsky, 20)

For any fixed $d \geq 1$ and $q_d \neq 0$

let $\omega_{g,n}^{\mathbf{q},[d]}$ be the outcome of TR for

$$\mathcal{S}^{\mathbf{q},[d]} : \begin{cases} x(z) & = & \ln z - \sum_{j=1}^{d} q_j z^j \\ y(z) & = & \sum_{j=1}^{d} q_j z^j \end{cases} \quad \text{with} \quad B = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}$$

We have the identity of series expansion near $e^{x_i} \to 0$

$$\omega_{g,n}^{\mathbf{q},[d]} - \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}(e^{x_1})\mathrm{d}(e^{x_2})}{\left(e^{x_1} - e^{x_2}\right)^2} \sim W_{g,n}(x_1, \ldots, x_n)|_{q_j = 0 \ j > d}$$

Strategy of the proof

- The difficult part of the proof is to justify quasipolynomiality wrt. $\nu_1, \dots, \nu_n$

$$\mathbb{H}_{g,n}(\nu_1, \dots, \nu_n) = \sum_{\substack{1 \leq a_1, \dots, a_n \leq d \\ m_1, \dots, m_n \geq 0}} C_{g,n}\begin{pmatrix} a_1 & \cdots & a_n \\ m_1 & \cdots & m_n \end{pmatrix} \prod_{i=1}^{n} A_{\nu_i, a_i} \nu_i^{m_i}$$

where $\quad A_{k,a} = \sum_{\lambda \vdash k-a} \frac{a\, k^{\ell(\lambda)}\, \vec{q}_\lambda}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} = [z^{k-a}] a\, e^{k \sum_{j=1}^{d} q_j z^j}$

for some $\quad C_{g,n}\begin{pmatrix} a_1 & \cdots & a_n \\ m_1 & \cdots & m_n \end{pmatrix} \in \mathbb{Q}(q_1, \dots, q_d)$ vanishing for $\sum_i m_i < M_{g,n}$

Done via combinatorial analysis of the formulas in $\mathcal{B} \cong \bigoplus_{L \geq 0} Z(\mathbb{Q}[\mathfrak{S}_L])$

- For the generating series, it implies the decomposition

$$\mathbb{W}_{g,n}(x_1, \dots, x_n) = \sum_{\substack{1 \leq a_1, \dots, a_n \leq d \\ m_1, \dots, m_n \geq 0}} C_{g,n}\begin{pmatrix} a_1 & \cdots & a_n \\ m_1 & \cdots & m_n \end{pmatrix} \prod_{i=1}^{n} \mathrm{d}\xi_{a_i, m_i} \quad \text{where} \quad \xi_{a,m} = \partial_{x(z)}^{m+1}(z^a)$$

hence it analytically continue to $(\mathcal{S}^{\mathbf{q},[d]})^n$ with poles at $z_i \to \mathfrak{a}$

- For the generating series, it implies the decomposition

$$\mathbb{W}_{g,n}(x_1,\ldots,x_n) = \sum_{\substack{1\leq a_1,\ldots,a_n\leq d \\ m_1,\ldots,m_n\geq 0}} C_{g,n}\left(\begin{smallmatrix} a_1 & \cdots & a_n \\ m_1 & \cdots & m_n \end{smallmatrix}\right) \prod_{i=1}^{n} \mathrm{d}\xi_{a_i,m_i} \quad \text{where} \quad \xi_{a,m} = \partial_{x(z)}^{m+1}(z^a)$$

hence it analytically continue to $(\mathcal{S}^{\mathbf{q},[d]})^n$ with poles at $z_i \to \mathfrak{a}$

- Then, we can use analytic continuation in the cut-and-join equations and combinatorial analysis similar to the simple or orbifold proofs

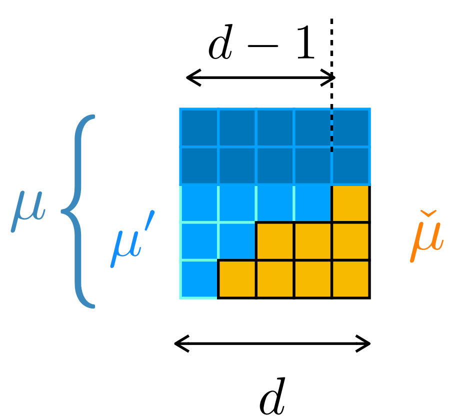
$\rightsquigarrow$ get a recursion for the polar part of $\mathbb{W}_{g,n}$ at $z_1 \to \mathfrak{a}$

solution by the topological recursion

Theorem 2 (B., Do, Karev, Lewanski, Moskowsky 20)

$$\mathbb{H}_{g,n}(\nu_1,\ldots,\nu_n) = \sum_{\mu\vdash|\nu|} d^{2g-2+n+\ell(\mu)}\vec{q}_\mu \prod_{i=1}^{n} \frac{\left(\frac{\nu_i}{d}\right)^{\lfloor \frac{\nu_i}{d}\rfloor}}{\lfloor \frac{\nu_i}{d}\rfloor !}$$

$$\times\left(\sum_{m=0}^{\ell(\mu')} \frac{(-1)^{\ell(\mu')-m}}{m!} \sum_{\substack{\rho\in\mathscr{P}_{d-1}^{m} \\ \sqcup_j \rho^{(j)} = \check{\mu}'}} \prod_{j=1}^{m} \frac{|C_{\rho^{(j)}}|\left[\frac{d-|\rho^{(j)}|}{d}\right]_{\ell(\rho^{(j)})-1}}{|\rho^{(j)}|!} \int_{\overline{\mathcal{M}}_{g,n+m}} \frac{\Omega^{(d,d)}_{g,n+m}\big(-\overline{\nu}_1,\ldots,-\overline{\nu}_n, d-|\rho^{(1)}|,\ldots,d-|\rho^{(m)}|\big)}{\prod_{i=1}^{n}\left(1-\frac{\nu_i}{d}\psi_i\right)}\right)$$



$\mathscr{P}_{d-1}$ = partitions of size at most $d-1$

$$[x]_b = x(x+1)\cdots(x+b-1)$$

Strategy of the proof : interpolation between the double and d-orbifold cases

$$\mathcal{S}_t \; : \; \begin{cases} x_t(z) & = & \ln z - q_d z^d - t \sum_{j=1}^{d-1} q_j z^j \\ y_t(z) & = & q_d z^d + t \sum_{j=1}^{d-1} q_j z^j \\ \omega_{0,2}^t & = & \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} \end{cases} \xrightarrow{\text{TR}} \omega_{g,n}^t$$

$t = 0$

$$\omega_{g,n}^0(z_1, \dots, z_n) = \omega_{g,n}^{[d]}(\tilde{z}_1, \dots, \tilde{z}_n)$$

$$\tilde{z} = q_d^{1/d} z \qquad x_0(z) = x_{[d]}(\tilde{z}) - \ln(q_d^{1/d})$$

$t = 1$

$$\omega_{g,n}^1 = \omega_{g,n}^{\mathbf{q},[d]}$$

Write the (convergent) Taylor expansion

$$\omega_{g,n}^1(z_1, \dots, z_n) = \sum_{l \geq 0} \frac{1}{l!} \partial_t^l \omega_{g,n}^t \omega_{g,n}(z_1, \dots, z_n)|_{t=0}$$

where all derivatives are taken at $x_i = x_t(z_i)$ fixed

as we want to extract the series expansion in variable $e^{x_i} \to 0$

II.3 Intersection theory and topological recursion — Double Hurwitz numbers

TR has the following property under deformations of initial data

$$\partial_t \omega_{g,n}^t(z_1,\dots,z_n) = \int_{z'\in\gamma} \Upsilon_t(z')\,\omega_{g,n+1}^t(z',z_1,\dots,z_n) \qquad \text{at fixed} \quad x_i = x_t(z_i)$$

whenever $\varpi_t(z) := \partial_t x_t(z)\mathrm{d}y_t(z) - \partial_t y(z)\mathrm{d}x_t(z) = \int_{z'\in\gamma} \Upsilon_t(z')\,\omega_{0,2}^t(z,z')$

with γ away from $\mathfrak{a}$

For the deformation we are interested in : $\int_{z'\in\gamma} \Upsilon_t(z') \bullet = -\operatorname*{Res}_{z'=\infty} \sum_{j=1}^{d-1} \frac{q_j}{j}\,(z')^j \bullet$

To compute higher order derivatives, need successive chain rules since $x_t(z') = x'$ is considered fixed

$\leadsto$ relates $\omega_{g,n}^{\mathbf{q},[d]}$ to $(\omega_{g,n+m}^{[d]})_{m\geq 0}$ and thus to Chiodo integrals

$\leadsto$ get the theorem by extracting the coefficients of $\prod_{i=1}^n \mathrm{d}(e^{\mu_i x_i})$

The resulting formula for $\mathbb{H}_{g,n}(\nu_1,\ldots,\nu_n)$ is a polynomial in $q_1,\ldots,q_{d-1}$ but only a Laurent polynomial in q_d

By its combinatorial meaning, it must be a polynomial in $q_1,\ldots,q_d$

$\Longrightarrow$ the Chiodo integral in prefactor of q_d^{-m} for $m>0$ must vanish

Corollary 3 (B., Do, Karev, Lewanski, Moskowsky 20)

Let $g\geq 0,\ n,\ell\geq 1\,,\ d\geq 2$

For any partitions $(\nu_1,\ldots,\nu_n),(\eta_1,\ldots,\eta_\ell)$ such that $\eta_1\leq d-1$ and $|\nu|+|\eta|<d\ell$

$$\sum_{k=1}^{\ell}\frac{(-1)^{\ell-k}}{k!}\sum_{\substack{\rho\in\mathscr{P}_{d-1}^{k}\\ \sqcup_a\rho^{(a)}=\eta}}\prod_{a=1}^{k}\frac{\left[\frac{d-|\rho^{(a)}|}{d}\right]_{\ell(\rho^{(a)})-1}}{\prod_j|\rho_j^{(a)}|!}\int_{\overline{\mathcal{M}}_{g,n+k}}\frac{\Omega_{g,n}^{(d,d)}\big(-\overline{\nu}_1,\ldots,-\overline{\nu}_n,d-|\rho^{(1)}|,\ldots,d-|\rho^{(k)}|\big)}{\prod_{i=1}^{n}(1-\frac{\nu_i}{d}\psi_i)}=0$$

Corollary 3 (B., Do, Karev, Lewanski, Moskowsky 20)

Let $g \geq 0,\ n, \ell \geq 1\ ,\ d \geq 2$

For any partitions $(\nu_1, \ldots, \nu_n), (\eta_1, \ldots, \eta_\ell)$ such that $\eta_1 \leq d-1$ and $|\nu| + |\eta| < d\ell$

$$\sum_{k=1}^{\ell} \frac{(-1)^{\ell-k}}{k!} \sum_{\substack{\rho \in \mathscr{P}_{d-1}^k \\ \sqcup_a \rho^{(a)} = \eta}} \prod_{a=1}^{k} \frac{\left[\frac{d-|\rho^{(a)}|}{d}\right]_{\ell(\rho^{(a)})-1}}{\prod_j |\rho_j^{(a)}|!} \int_{\overline{\mathcal{M}}_{g,n+k}} \frac{\Omega_{g,n}^{(d,d)}\big(-\overline{\nu}_1, \ldots, -\overline{\nu}_n, d-|\rho^{(1)}|, \ldots, d-|\rho^{(k)}|\big)}{\prod_{i=1}^n (1 - \frac{\nu_i}{d}\psi_i)} = 0$$

- When $\eta_i + \eta_j \geq d$ for any $i \neq j$, there is a single term, and then

$$\int_{\overline{\mathcal{M}}_{g,n+\ell}} \frac{\Omega_{g,n}^{(d,d)}\big(-\overline{\nu}_1, \ldots, -\overline{\nu}_n, d-\eta_1, \ldots, d-\eta_\ell\big)}{\prod_{i=1}^n \left(1 - \frac{\nu_i}{d}\psi_i\right)} = 0$$

  This vanishing is proved by Johnson-Pandharipande-Tseng 11 from geometry of $\overline{\mathcal{M}}_{g,a_1,\ldots,a_n}(B\mathbb{Z}_d)$

- Other cases are new
  (would require a careful analysis of the boundary strata contributions in JPT)

Tested numerically with the SAGE package *Admcycles*

Delecroix, Schmitt, van Zelm
2002.01709

Double Hurwitz numbers

$H_{1,1}(2)$	$[q_1^2]$	$\frac{1}{12}$	$\frac{27}{2}\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[3]}_{1;1,1,1}}{1-2\psi_1/3}$
	$[q_2]$	$\frac{1}{4}$	$9\int_{\overline{\mathcal{M}}_{1,2}} \frac{\Omega^{[3]}_{1;1,2}}{1-2\psi_1/3}$
$H_{1,1}(3)$	$[q_1^3]$	$\frac{3}{8}$	$\frac{27}{2}\int_{\overline{\mathcal{M}}_{1,4}} \frac{\Omega^{[3]}_{1;0,1,1,1}}{1-\psi_1}$
	$[q_1q_2]$	$\frac{3}{2}$	$27\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[3]}_{1;0,1,2}}{1-\psi_1}$
	$[q_3]$	1	$3\int_{\overline{\mathcal{M}}_{1,1}} \frac{\Omega^{[3]}_{1;0}}{1-\psi_1}$
$H_{1,1}(4)$	$[q_1^4]$	$\frac{4}{3}$	$\frac{27}{2}\int_{\overline{\mathcal{M}}_{1,5}} \frac{\Omega^{[3]}_{1;2,1,1,1,1}}{1-4\psi_1/3}$
	$[q_1^2q_2]$	$\frac{20}{3}$	$54\int_{\overline{\mathcal{M}}_{1,4}} \frac{\Omega^{[3]}_{1;2,1,1,2}}{1-4\psi_1/3}$
	$[q_2^2]$	$\frac{7}{3}$	$18\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[3]}_{1;2,2,2}}{1-4\psi_1/3} - 6\int_{\overline{\mathcal{M}}_{1,2}} \frac{\Omega^{[3]}_{1;2,1}}{1-4\psi_1/3}$
	$[q_1q_3]$	6	$36\int_{\overline{\mathcal{M}}_{1,2}} \frac{\Omega^{[3]}_{1;2,1}}{1-4\psi_1/3}$
$H_{1,1}(5)$	$[q_1^5]$	$\frac{625}{144}$	$\frac{81}{8}\int_{\overline{\mathcal{M}}_{1,6}} \frac{\Omega^{[3]}_{1;1,1,1,1,1,1}}{1-5\psi_1/3}$
	$[q_1^3q_2]$	$\frac{625}{24}$	$\frac{135}{2}\int_{\overline{\mathcal{M}}_{1,5}} \frac{\Omega^{[3]}_{1;1,1,1,1,2}}{1-5\psi_1/3}$
	$[q_1q_2^2]$	$\frac{125}{6}$	$\frac{135}{2}\int_{\overline{\mathcal{M}}_{1,4}} \frac{\Omega^{[3]}_{1;1,1,2,2}}{1-5\psi_1/3} - \frac{45}{2}\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[3]}_{1;1,1,1}}{1-5\psi_1/3}$
	$[q_1^2q_3]$	$\frac{625}{24}$	$\frac{135}{2}\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[3]}_{1;1,1,1}}{1-5\psi_1/3}$
	$[q_2q_3]$	$\frac{25}{2}$	$45\int_{\overline{\mathcal{M}}_{1,2}} \frac{\Omega^{[3]}_{1;1,2}}{1-5\psi_1/3}$

Vanishing identities

- $$\frac{1}{2}\int_{\overline{\mathcal{M}}_{1,2}} \frac{\Omega^{[6]}_{1;3,3}}{\left(1-\frac{\psi_1}{2}\right)} = \int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[6]}_{1;3,4,5}}{\left(1-\frac{\psi_1}{2}\right)}. \qquad = \frac{1}{72}$$
- $$\int_{\overline{\mathcal{M}}_{1,4}} \frac{\Omega^{[4]}_{1;2,1,2,3}}{\left(1-\frac{\psi_1}{2}\right)} = \frac{1}{4}\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[4]}_{1;2,1,1}}{\left(1-\frac{\psi_1}{2}\right)}. \qquad = \frac{1}{1536}$$
- $$\frac{1}{2}\int_{\overline{\mathcal{M}}_{1,4}} \frac{\Omega^{[7]}_{1;5,5,5,6}}{\left(1-\frac{2}{7}\psi_1\right)} + \frac{1}{2}\cdot\frac{2}{7}\cdot\frac{9}{7}\int_{\overline{\mathcal{M}}_{1,2}} \frac{\Omega^{[7]}_{1;5,2}}{\left(1-\frac{2}{7}\psi_1\right)}$$
$$= \frac{3}{7}\cdot\frac{1}{2}\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[7]}_{1;5,6,3}}{\left(1-\frac{2}{7}\psi_1\right)} + \frac{4}{7}\int_{\overline{\mathcal{M}}_{1,3}} \frac{\Omega^{[7]}_{1;5,5,4}}{\left(1-\frac{2}{7}\psi_1\right)}.$$

Thank you for your attention !

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