Resolution of Liouville CFT: Segal axioms and bootstrap

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2d Conformal Field Theory (CFT)

A Conformal Field Theory is a quantum field theory with extra symmetries (conformal invariance).

1) Objects:

- M compact surface equipped with Riemannian metric g, and conformal class

$$[g] := \{e^{\omega} g \,|\, \omega \in C^{\infty}(M)\}.$$

- A classical action $S_g : E \to R$ (E =space of fields , e.g. $E = H^s(M)$ for some $s \in \mathbb{R}$), depending on the background metric g. For example: Dirichlet energy (free field theory)

$$S_g(\varphi) := \int_M |\nabla^g \varphi(x)|_g^2 dv_g(x)$$

Physicists consider special quantities represented by Feynmann path integrals:

2) Correlation and partition functions (in physics):

- Partition fct: the mass of the formal measure $e^{-S_g(\varphi)}D\varphi$ on E

$$Z_g := \int_E e^{-S_g(\varphi)}D\varphi$$

- Correlation fct: $x_1,\dots,x_n \in M$ some points, $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ some weights

$$\langle V_{\alpha_1}(x_1)\dots V_{\alpha_n}(x_n)\rangle_g := \int_E e^{\alpha_1\varphi(x_1)}\dots e^{\alpha_n\varphi(x_n)}e^{-S_g(\varphi)}D\varphi$$

$V_{\alpha_i}(x_i) = e^{\alpha_i\varphi(x_i)}$ are called insertions at x_i with weights α_i.

Math definition of CFT

A Conformal Field Theory in dim $= 2$ is the data for each Riemannian closed surface (M, g) of

- partition function $Z_g \in \mathbb{R}$
- Correlation functions $\langle V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n)\rangle_g \in \mathbb{R}$

satisfying the properties:

- Diffeomorphism invariance: for any $\psi : M \to M$ orientation preserv. diffeo,

$$Z_{\psi^* g} = Z_g, \quad \langle V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n)\rangle_{\psi^* g} = \langle V_{\alpha_1}(\psi(x_1)) \dots V_{\alpha_n}(\psi(x_n))\rangle_g$$

- Conformal covariance: if $\omega \in C^\infty$ and $\hat{g} = e^\omega g$

$$Z_{\hat{g}} = Z_g \exp\Big(\frac{\mathbf{c}}{96\pi} \int_M |d\omega|_g^2 + 2K_g \omega\Big)$$

$$\langle V_{\alpha_1}(x_N) \dots V_{\alpha_n}(x_n)\rangle_{\hat{g}} = \langle V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n)\rangle_g \exp\Big(-\sum_{j=1}^n \Delta_{\alpha_k} \omega(x_k)\Big) \frac{Z_{\hat{g}}}{Z_g}$$

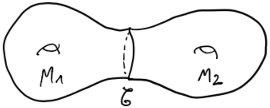
K_g =scalar curv. of g, $(\Delta_\alpha, \mathbf{c})$=constants associated to the CFT.

Main problem in physics: Since (a priori), Feynmann integral is not mathematically defined, guess expressions for what should be Z_g and $\langle V_{\alpha_1}(x_1)\dots V_{\alpha_n}(x_n)\rangle_g$.

Tools in physics: Use conformal symmetries. The axioms impose strong restriction on the possible values of partition and correlations fcts:

- Integrability $\Longrightarrow (x_1,\dots,x_n)\mapsto \langle V_{\alpha_1}(x_1)\dots V_{\alpha_n}(x_n)\rangle_g$ should satisfy certain PDE
- Decomposition into blocks: A Feynmann integral can be decomposed into Feynmann integrals over fields restricted to subdomains of the surface M (Segal axioms), this method is related to the conformal bootstrap (dev. by Belavin-Polyakov-Zamolodchikov ('84))

Segal axioms (physics heuristics)



Desintegration of Feynmann integral using conditionning on $\mathcal{C} = \partial M_1 = \partial M_2$: if

$$S_{M,g}(\varphi) = S_{M_1,g}(\varphi|_{M_1}) + S_{M_2,g}(\varphi|_{M_2})$$

one should have

$$\int_{E(M)} e^{-S_{M,g}(\varphi)} D\varphi = \int_{E(\mathcal{C})} \Big(\int_{\substack{E(M_1), \\ \varphi=\varphi_0}} e^{-S_{M_1,g}(\varphi|_{M_1})} D\varphi \Big) \Big(\int_{\substack{E(M_2), \\ \varphi=\varphi_0}} e^{-S_{M_2,g}(\varphi|_{M_2})} D\varphi \Big) D\varphi_0$$

$$= \int_{E(\mathcal{C})} \mathcal{A}_{M_1}(\varphi_0) \mathcal{A}_{M_2}(\varphi_0) D\varphi_0$$

Segal axioms

A CFT is

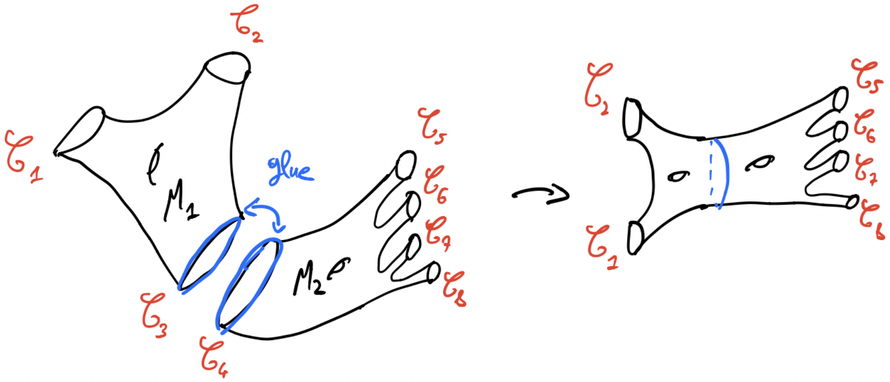
- Object: $\mathcal{H}$ a Hilbert space attached to $\mathbb{S}^1$ (where $\mathcal{H}$ should represent $L^2(E(\mathbb{S}^1))$)
- Morphism: to each (M,g) Riemannian surface with parametrized geodesic boundary $\partial M = \sqcup_{i=1}^{b} \mathcal{C}_i$, we associate an amplitude $\mathcal{A}_{M,g} \in \otimes^b \mathcal{H}$
- Conformal covariance: for $\omega \in C^\infty(M)$ with $\omega = 0$ on ∂M

$$\boxed{\mathcal{A}_{M,e^\omega g} = e^{\frac{c}{96\pi}\int_M |d\omega|_g^2 + 2K_g\omega} \mathcal{A}_{M,g}}$$

- Gluing/fonctoriality: if we glue (M_1,g_1) with (M_2,g_2) by identifying $\mathcal{C}_{j_1} \sim \mathcal{C}_{j_2}$ $(\partial M_1 = \sqcup_{j=1}^{b_1}\mathcal{C}_j$, and $\partial M_2 = \sqcup_{j=b_1+1}^{b_1+b_2}\mathcal{C}_j)$, for $(M,g) := (M_1 \sharp M_2, g_1 \sharp g_2)$

$$\boxed{\mathcal{A}_{M,g} = \mathrm{Tr}_{j_1 j_2}(\mathcal{A}_{M_1,g_1} \otimes \mathcal{A}_{M_2,g_2})}$$

where Tr_{ij} is the ij partial trace on $\otimes^{b_1+b_2}\mathcal{H}$ (maps to $\otimes^{b_1+b_2-2}\mathcal{H}$).



$$\mathcal{A}_{M,g}(\Phi_1\,\Phi_2,\Phi_5,\Phi_6,\Phi_7,\Phi_9)=\langle\underbrace{\mathcal{A}_{M_1,g_1}(\Phi_1,\Phi_2,\cdot)}_{\in\mathcal{H}},\underbrace{\mathcal{A}_{M_2,g_2}(\cdot,\Phi_5,\Phi_6,\Phi_7,\Phi_9)}_{\in\mathcal{H}}\rangle_{\mathcal{H}}$$

Liouville action on Riemannian surface (M, g) is

$$\boxed{S_g(\varphi) = \frac{1}{4\pi}\int_M (|d\varphi|_g^2 + QK_g\varphi + e^{\gamma\varphi})\mathrm{dv}_g}$$

with $Q = 2/\gamma + \gamma/2$ and $\gamma \in (0, 2)$, K_g =scalar curvature of g

- In physics, $Q := 2/\gamma + \gamma/2$ with $\gamma \in (0, 2]$ and Liouville CFT is a CFT with central charge and conformal weights

$$\mathbf{c} = 1 + 6Q^2, \quad \Delta_\alpha := \frac{\alpha}{2}(Q - \frac{\alpha}{2})$$

- Critical points of S_g are related to finding φ_0 s.t. $K_{e^{\gamma\varphi_0}g}$ = negative constant.

Mathematical definition of Liouville CFT

- $(\Omega, \mathbb{P})$ proba space, (α_n) i.i.d Gaussian in $\mathcal{N}(0,1)$
- Gaussian Free Field $X_g := \sqrt{2\pi}\sum_{n>0}\alpha_n \frac{u_n}{\sqrt{\lambda_n}} \in H^{-\varepsilon}(M)$ where $(\lambda_n)_{n=0}^{\infty}$ spectrum of Δ_g with eigenfunctions u_n:

$$\mathbb{E}[X_g(x)X_g(x')] = 2\pi\Delta_g^{-1}(x,x') \sim -\log(d_g(x,x')) + C^0(M\times M)$$

- There is a measure $\mathcal{P}'$ on $H^{-\varepsilon}(M)\cap 1^{\perp}$ which is the law of the random variable X_g, then set $\mathcal{P} = dc \otimes \mathcal{P}'$ on $H^{-\varepsilon}(M)$, dc = Lebesgue on $\mathbb{R} = \ker \Delta_g$.
- This allows to define the Gaussian integral: for $F \in L^1(H^{-\varepsilon}(M), \mathcal{P})$

$$\int_{H^{-\varepsilon}(M)} F(\varphi)e^{-\frac{1}{4\pi}\int_M |d\varphi|^2} D\varphi \stackrel{def}{=} \frac{\sqrt{\mathrm{Vol}_g(M)}}{\sqrt{\det'(\Delta_g)}}\int_{\mathbb{R}} \mathbb{E}[F(c+X_g)]dc.$$

Define the correlations/partition functions by the probabilistic expression:

$$\langle V_{\alpha_1}(x_1)\dots V_{\alpha_n}(x_n)\rangle_g \stackrel{def}{=} \frac{\sqrt{\mathrm{Vol}(M)}}{\sqrt{\det'(\Delta_g)}}\int_{\mathbb{R}} \mathbb{E}\Big[\prod_{j=1}^{n} e^{\alpha_i(c+X_g(x_i))} e^{-\frac{1}{4\pi}\int_M (QK_g(c+X_g)+e^{\gamma(c+X_g)})\mathrm{dv}_g}\Big]dc$$

Kahane '85: Gaussian multiplicative chaos: one can make sense of the random measure $e^{\gamma X_g}\mathrm{dv}_g$ for $\gamma\in(0,2)$: convolve X_g at small scale $\delta>0$ and let

$$e^{\gamma X_g}\mathrm{dv}_g \stackrel{def}{=} \lim_{\delta\to 0} e^{\gamma X_{g,\delta}-\frac{\gamma^2}{2}\mathbb{E}[X_{g,\delta}^2]}\mathrm{dv}_g$$

Condition for existence: $\sum_i \alpha_i - Q\chi(M)>0$ and $\alpha_i<Q$, called Seiberg bounds.

Theorem (David-Kupiainen-Rhodes-Vargas '16, D-R-V'16, Guillarmou-R-V '18)

Liouville CFT constructed with probability is a CFT with central charge $\mathbf{c}_L=1+6Q^2$ for $\gamma\in(0,2]$ and conformal weights $\Delta_\alpha:=\frac{\alpha}{2}(Q-\frac{\alpha}{2})$.

Hilbert space of LCFT (Segal axioms)

Hilbert space of LCFT: if $\Omega := (\mathbb{R}^2)^{\mathbb{N}^*}$ and $\mathbb{P} = \prod_{n\geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2+y_n^2)} dx_n dy_n$,

$$\boxed{\mathcal{H} := L^2(\mathbb{R}_c \times \Omega, dc \otimes \mathbb{P}) = L^2(H^{-\varepsilon}(\mathbb{S}^1), d\mu)}$$

where μ is pushfoward of $dc \otimes \mathbb{P}_\Omega$ by the random field $\varphi = c + \frac{1}{2} \sum_{n\neq 0} \frac{x_n + iy_n}{|n|^{1/2}} e^{in\theta}$.

Definition of amplitudes (Segal axioms)

Probabilistic definition of amplitude of (M,g) with b parametrized geodesic boundary circles and n weighted marked points (x_i,α_i):

$$\boxed{\mathcal{A}_{M,g,x,\alpha}(\varphi) \stackrel{def}{=} \mathbb{E}\Big[\prod_{i=1}^n e^{\alpha_i(X_D(x_i)+P\varphi(x_i))} e^{-\frac{1}{4\pi}\int_M (QK_g(X_D+P\varphi)+e^{\gamma(X_D+P\varphi)})\mathrm{dv}_g}\Big]\,\mathcal{A}^0_{M,g}(\varphi)},$$

- $c + X_g = X_D + P\varphi$ with $\varphi = (\varphi^1,\dots,\varphi^b) \in H^{-\varepsilon}(\mathbb{S}^1)^b$,
- X_D = GFF with Dirichlet condition, $\mathbb{E}$ = expectation wrt X_D,
- $P\varphi$ = harmonic extension of φ on M with

$$\varphi^j = c^j + \sum_{n\neq 0} \varphi^j_n e^{in\theta}, \quad \varphi^j_n = \frac{x^j_n + iy^j_n}{2\sqrt{n}}, \quad x^j_n, y^j_n \in \mathcal{N}(0,1), \quad c^j \in \mathbb{R}.$$

- $\mathcal{A}^0_{M,g}(\varphi) = e^{-\frac{1}{2}\langle (D_M - D)\varphi,\varphi\rangle}$ half-density term (D_M=DN map on M, $D = \sqrt{\Delta_{\mathbb{S}^1}}$).

Segal Axioms for LCFT

Recall $\mathcal{H} := L^2(H^{-\varepsilon}(\mathbb{S}^1), d\mu)$.

Theorem (G-Kupiainen-Rhodes-Vargas '21)

1) Let (M, g) be Riemannian surface with b parametrized geodesic boundary circles, marked points $x = (x_1, \ldots, x_m)$ with weight $\alpha = (\alpha_1, \ldots, \alpha_m)$. Then if $\sum_i \alpha_i + Q\chi(M) > 0$

$$\mathcal{A}_{M,g,x,\alpha} \in L^2(H^{-\varepsilon}(\mathbb{S}^1)^b, d\mu^b) = \mathcal{H}^{\otimes b}.$$

2) The amplitudes satisfy conformal covariance required in Segal axioms.
3) The amplitudes satisfy gluing properties required in Segal axioms.

DOZZ and Conformal bootstrap

1) 3-pt function on $\mathbb{S}^2$: Using Möbius transform $\psi \in \mathrm{PSL}_2(\mathbb{C})$ and conformal covariance, 3-point function on $\mathbb{S}^2$ reduces to knowing $C(\alpha_1,\alpha_2,\alpha_3) := \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\mathbb{S}^2}$

Kupiainen-Rhodes-Vargas '17: There is an explicit formula for the 3pt function $C(\alpha_1,\alpha_2,\alpha_3)$ in terms of special functions, called DOZZ formula (Dorn-Otto-Zamolodchikov-Zamolodchikov)

Let $(x_1,x_2,x_3)=(0,1,\infty)$, $\bar{\alpha} := \alpha_1+\alpha_2+\alpha_3$, then for $g=\frac{|dz|^2}{\max(1,|z|^4)}$ on $\hat{\mathbb{C}}\simeq\mathbb{S}^2$

$$\langle\prod_{i=1}^{3} e^{\alpha_i\varphi(x_i)}\rangle_{\mathbb{S}^2} := \Big(\frac{\pi\mu\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\,\big(\frac{\gamma}{2}\big)^{2-\frac{\gamma^2}{2}}\Big)^{\frac{2Q-\bar{\alpha}}{\gamma}}\frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-Q)\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-\alpha_3)}$$

$$\text{with } \Upsilon_{\frac{\gamma}{2}}(z) := \exp\Big(\int_0^\infty\Big(\Big(\frac{Q}{2}-z\Big)^2 e^{-t}-\frac{(\sinh((\frac{Q}{2}-z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})}\Big)\frac{dt}{t}\Big).$$

2) Conformal bootstrap: use representation theory to express n-pt correlation function in terms of 3-pt functions and universal functions called conformal blocks depending on $(\mathbf{c}, \Delta_\alpha)$.

Theorem (G-Kupiainen-Rhodes-Vargas '20)

The 4-pt function on the sphere is given in terms of 3-pt functions and conformal blocks $\mathcal{F}_{p,\alpha}(z)$ by: if $(x_1, x_2, x_3, x_4) = (0, z, 1, \infty)$ with $|z| < 1$ and $\alpha = (\alpha_1, \dots, \alpha_4)$

$$\langle \prod_{i=1}^{4} V_{\alpha_i}(x_i) \rangle_{\mathbb{S}^2} = \frac{1}{2\pi} \int_0^\infty \overbrace{C(\alpha_1, \alpha_2, Q - ip)}^{\text{3-pt fct}} \overbrace{C(Q + ip, \alpha_3, \alpha_4)}^{\text{3-pt fct}} |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_{p,\alpha}(z)|^2 dp$$

Heuristic

Heuristically, this can be thought of as

$$\int_0^\infty \Big\langle V_{\alpha_1}(0)V_{\alpha_2}(z), V_{Q+ip}(\infty)\Big\rangle_{\mathcal{H}}\Big\langle V_{Q+ip}(\infty), V_{\alpha_3}(1)V_{\alpha_4}(0)\Big\rangle_{\mathcal{H}}|z|^{a(p,\alpha)}|\mathcal{F}_{p,\alpha}(z)|^2dp$$

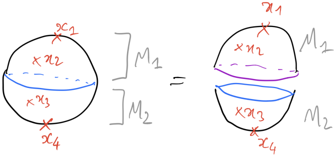
where we think of 3-pt function as a scalar product $\langle V_{\alpha_1}(0)V_{\alpha_2}(z), V_{Q+ip}(\infty)\rangle_{\mathcal{H}}$ in a Hilbert space $\mathcal{H}$ of two states (or amplitudes) $u_1 = V_{\alpha_1}(0)V_{\alpha_2}(z)$ and $u_2 = V_{Q+ip}(\infty)$.

This ressembles a Plancherel formula where $(V_{Q+ip}(\infty))_{p>0}$ form a basis of eigenfunctions of an operator H on $\mathcal{H}$.

$$\langle f_1, f_2\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}}\langle f_1, e^{ipc}\rangle\langle e^{ipc}, f_2\rangle dp, \quad -\partial_c^2 e^{ipc} = p^2 e^{ipc}.$$

Conformal bootstrap for 4-point function on $\mathbb{S}^2$

Idea: cut the sphere in two disks M_1, M_2:



Let $\varphi := c + \frac{1}{2}\sum_{n\neq 0} |n|^{-1/2}(x_n + iy_n)e^{in\theta} \in H^{-\varepsilon}(\mathbb{S}^1)$ real-valued random field with $x_n, y_n \in \mathcal{N}(0,1)$ i.i.d., and define for $i = 1, 2$ the amplitudes

$$A_{M_i,g}(\varphi) = \mathbb{E}\Big[\prod_{j\leq 2i}^{2} e^{\alpha_j(X_D+P\varphi)(x_j)} e^{-\frac{1}{4\pi}\int_{M_i}(QK_g(X_D+P\varphi)+e^{\gamma(X_D+P\varphi)})\mathrm{dv}_g}\Big]$$

- X_D = GFF on M_i with Dirichlet condition, $\mathbb{E}$ = expectation wrt X_D,
- $P\varphi$ harmonic extension of φ: $\Delta_{M_i}P\varphi = 0$, $P\varphi|_{\partial M_i} = \varphi$

Corollary (Gluing formula)

The amplitude of M_i is in $L^2(H^{-\varepsilon}(\mathbb{S}^1)^b, d\mu) = \mathcal{H}$ and

$$\langle \prod_{j=1}^{4} V_{\alpha_j}(x_j) \rangle_{\mathbb{S}^2} = \int_{H^{-\varepsilon}(\mathbb{S}^1)} A_{M_1}(\varphi) A_{M_2}(\varphi) d\mu(\varphi) = \langle A_{M_1}, A_{M_2} \rangle_{\mathcal{H}}$$

Tool for proof: decompose GFF on $\mathbb{S}^2$ as $X_g = X_{M_1,D} + X_{M_2,D} + P\varphi$ ($\simeq$conditioning on $\mathbb{S}^1$)

Idea: decompose the $\mathcal{H}$ pairing using diagonalization of a certain operator (Plancherel formula).

The propagator and the Hamiltonian

For the flat annulus $\mathbb{A}_t = (\{z \in \mathbb{C} \,|\, e^{-t} \leq |z| \leq 1\}, g = \frac{|dz|^2}{|z|^2})$, define the amplitude as above

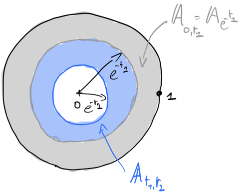
$$A_{\mathbb{A}_t}(\varphi, \varphi') := \mathbb{E}\Big[e^{-\frac{1}{4\pi}\int_{\mathbb{A}_t} e^{\gamma(X_D + P(\varphi,\varphi'))}\mathrm{dv}_g}\Big] e^{-\frac{1}{2}\langle(\mathrm{D}_{\mathbb{A}_t} - \mathrm{D})(\varphi,\varphi'),(\varphi,\varphi')\rangle}$$

where $\mathrm{D}_{\mathbb{A}_t}$ =Dirichlet-to-Neumann of $\mathbb{A}_t$ and $\mathrm{D} = |\partial_\theta|$ (note: $\mathrm{D}_{\mathbb{A}_t} - \mathrm{D}$ is smoothing).

Define the associated operator $S(t) : \mathcal{H} \to \mathcal{H}$:

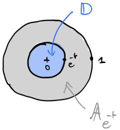
$$\forall \varphi \in H^{-\varepsilon}(\mathbb{S}^1), \quad \boxed{(S(t)F)(\varphi) := \int_{H^{-\varepsilon}(\mathbb{S}^1)} A_{\mathbb{A}_t}(\varphi, \varphi')F(\varphi')d\mu(\varphi')}$$

idea 1: gluing two annuli produces bigger annuli $\Longrightarrow S(t)$ should be a semi-group.



with $\mathbb{A}_{t_1,t_2} = \{|z| \in [e^{-t_2}, e^{-t_1}]\}$

idea 2: gluing annulus $\mathbb{A}_t$ with a disk $\mathbb{D}$ produces a bigger disk $\Longrightarrow S(t)A_{\mathbb{D},0,\alpha} = e^{\lambda t}A_{\mathbb{D},0,\alpha}$.



Proposition (G-Kupiainen-Rhodes-Vargas '20)

The operator $e^{-(\frac{1+6Q^2}{12})t}S(t) = e^{-t\mathrm{H}}$ *is a contraction semi-group on* $\mathcal{H} = L^2(\mathbb{R}\times\Omega; dc\otimes\mathbb{P})$ *with self-adjoint generator*

$$\boxed{\mathrm{H} = \frac{1}{2}(-\partial_c^2 + Q^2 + 2\mathrm{P} + e^{\gamma c}V) =: \mathrm{H}_0 + \frac{1}{2}e^{\gamma c}V}$$

with P *the infinite harmonic oscillator and* $V \in L^{\frac{2}{\gamma^2}-}(\Omega)$ *a positive potential/measure:*

$$\mathrm{P} := \sum_{n=1}^{\infty} n[(\partial_{x_n})^*\partial_{x_n} + (\partial_{y_n})^*\partial_{y_n}], \quad V(\tilde{\varphi}) := \frac{1}{2\pi}\int_{\mathbb{S}} e^{\gamma\tilde{\varphi}(\theta)}d\theta$$

where $\tilde{\varphi} = \varphi - \frac{1}{2\pi}\int_{\mathbb{S}^1}\varphi(\theta)d\theta = \varphi - c$.

Tool: Feynmann-Kac representation of $e^{-t\mathrm{H}}$.

Spectral resolution for the free field Hamiltonian H_0

Fact 1: P is self-adjoint on $L^2(\Omega, \mathbb{P})$ and has discrete spectrum $\sigma(\mathrm{P}) = \mathbb{N}_0$. Eigenfunctions are indexed by finite sequences $\mathrm{k} = (k_1, \dots, k_n, 0, \dots), \mathrm{l} = (l_1, \dots, l_{n'}, 0, \dots) \in \mathbb{N}^{\mathbb{N}}$ and given by

$$\psi_{\mathrm{kl}} = \prod_n h_{k_n}(x_n) h_{l_n}(y_n), \quad \mathrm{P}\psi_{\mathrm{kl}} = (|\mathrm{k}| + |\mathrm{l}|)\psi_{\mathrm{kl}}$$

with $h_k(x)$ (L^2-normalized) Hermite polynomial and $|\mathrm{k}| = \sum_n n k_n \in \mathbb{N}$.

Fact 2: $-\partial_c^2 + Q^2$ had continuous spectrum $\sigma(-\partial_c^2 + Q^2) = [Q^2, \infty)$, eigenfunctions are e^{ipc} with eigenvalue $p^2 + Q^2$.

Plancherel formula: for $u_1, u_2 \in L^2(\mathbb{R} \times \Omega)$

$$\boxed{\langle u_1, u_2 \rangle_{\mathcal{H}} = \sum_{\mathrm{k},\mathrm{l} \in \mathcal{N}} \int_{\mathbb{R}} \langle u_1, e^{ipc}\psi_{\mathrm{kl}} \rangle_{\mathcal{H}} \langle e^{ipc}\psi_{\mathrm{kl}}, u_2 \rangle_{\mathcal{H}} \, dp}$$

Diagonalization of H using scattering theory:

Theorem (G-Kupiainen-Rhodes-Vargas '20)

Let $\gamma \in (0,2)$, $Q = 2/\gamma + \gamma/2$. *Then*

- *The spectrum of* H *is absolutely continuous, each* $E \in [\frac{Q^2}{2}, \infty)$ *is of finite multiplicity*
- $\exists$ *a complete family of generalized eigenstates* $\Phi_{Q+ip,\mathrm{k},\mathrm{l}} \in \cap_{\varepsilon>0} e^{-\varepsilon c_-} L^2(\mathbb{R}_c \times \Omega)$ *labeled by* $p \in \mathbb{R}_+$ *and* $\mathrm{k} = (k_1, \dots, k_n, 0, \dots), \mathrm{l} = (l_1, \dots, l_{n'}, 0, \dots) \in \mathbb{N}^{\mathbb{N}}$ *s.t.*

$$\mathrm{H}\Phi_{Q+ip,\mathrm{k},\mathrm{l}} = \Big(\frac{Q^2}{2} + \frac{p^2}{2} + |\mathrm{k}| + |\mathrm{l}|\Big)\Phi_{Q+ip,\mathrm{k},\mathrm{l}}.$$

- $\Phi_{Q+ip,\mathrm{k},\mathrm{l}}$ *is a complete family diagonalizing* H: $\forall u_1, u_2 \in L^2(\mathbb{R} \times \Omega)$

$$\boxed{\langle u_1, u_2\rangle_{L^2} = \frac{1}{2\pi}\sum_{\mathrm{k},\mathrm{l}\in\mathcal{N}} \int_0^\infty \langle u_1, \Phi_{Q+ip,\mathrm{k},\mathrm{l}}\rangle_{L^2} \langle \Phi_{Q+ip,\mathrm{k},\mathrm{l}}, u_2\rangle_{L^2}\, dp}$$

Link with primary field and highest weight vector

Proposition (G-Kupiainen-Rhodes-Vargas'20)

1) The (probabilistic) amplitude of the unit disk $(\mathbb{D}, |dz|^2)$ *with insertion* $V_\alpha(0) = e^{\alpha\varphi(0)}$ *for* $\alpha < Q$*:*

$$\Phi_\alpha(\varphi) := A_{\mathbb{D},\alpha}(\varphi) = \mathbb{E}\Big[e^{\alpha(X_D + P\varphi)(0)} e^{-\frac{1}{4\pi}\int_{\mathbb{D}} e^{\gamma(X_D + P\varphi)} \mathrm{dv}_{\mathbb{D}}}\Big] \in e^{(\alpha - Q - \varepsilon)c_-} L^2(\mathbb{R} \times \Omega)$$

is an eigenfunction of H*:*

$$\boxed{H\Phi_\alpha = \alpha(Q - \frac{\alpha}{2})\Phi_\alpha = 2\Delta_\alpha \Phi_\alpha}$$

2) The map $\alpha \mapsto \Phi_\alpha$ *extends analytically to* $\mathrm{Re}(\alpha) \leq Q$ *with value in weighted space* $e^{(\alpha - Q - \varepsilon)c_-} L^2(\mathbb{R} \times \Omega)$ *and* $\Phi_{Q+ip,0,0} = \Phi_{Q+ip}$.

3) The map $\alpha \mapsto \Phi_{\alpha,\mathrm{k},\mathrm{l}}$ *also extends analytically to* $\{\mathrm{Re}(\alpha) \leq Q\} \cap \{|Q - \alpha| > |\mathrm{k}| + |\mathrm{l}|\}$

Remark: no probabilistic representation of Φ_α for $|\mathrm{Re}(\alpha - Q)| < |\mathrm{Im}(\alpha - Q)|$

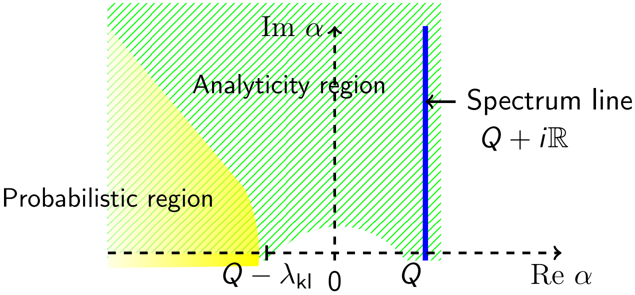


Figure: Analytic continuation of eigenstates $\Psi_{\alpha,\mathsf{k},\mathsf{l}}$ and probabilistic region, $\lambda_{\mathsf{kl}} := |\mathsf{k}| + |\mathsf{l}|$.

In probabilistic region, intertwining (scattering):

$$\Phi_{\alpha,\mathsf{k},\mathsf{l}} = \lim_{t\to\infty} e^{t(\Delta_\alpha + |\mathsf{k}|+|\mathsf{l}|)} e^{-t\mathsf{H}} (\underbrace{e^{(\alpha-Q)c}\psi_{\mathsf{kl}}}_{\mathsf{H}_0 \text{ eigenst}}).$$

Descendent fields and unitary representation of Virasoro algebra

- There is a family of operators $L_n, \tilde{L}_n$ on $L^2(\mathbb{R}\times\Omega)=L^2(H^{-\varepsilon}(\mathbb{S}^1))$ for $n\in\mathbb{Z}$ such that

$$L_n^*=L_{-n},\quad [L_n,L_m]=(n-m)L_{n+m}+\frac{\boldsymbol{c}}{12}(n^3-n)\delta_{n+m}$$

  and same for $\tilde{L}_n$.
- For $n>0$, L_n are annihilation operators, L_{-n} are creation operators
- $\mathrm{H}=L_0+\tilde{L}_0$ and

$$L_n\Phi_\alpha=\tilde{L}_n\Phi_\alpha=0,\quad \forall n>0$$

- For two Young diagrams $\nu=(\nu_1\geq\cdots\geq\nu_k)$ and $\tilde{\nu}=(\tilde{\nu}_1\geq\cdots\geq\tilde{\nu}_{\tilde{k}})$, $(\nu_j\in\mathbb{N})$

$$\Psi_{\alpha,\nu,\tilde{\nu}}:=\prod_{\ell,\tilde{\ell}}L_{-\nu_\ell}\tilde{L}_{-\tilde{\nu}_{\tilde{\ell}}}\Phi_\alpha$$

  is a linear combination of the $\Phi_{\alpha,\mathrm{k},\mathrm{l}}$ with $|\nu|+|\nu'|=|\mathrm{k}|+|\mathrm{l}|$.
- $(\Psi_{\alpha,\nu,\tilde{\nu}})_{\alpha,\nu,\tilde{\nu}}$ is not orthonormal

Spectral resolution in terms of Virasoro descendents, application

$$\boxed{\langle u_1, u_2\rangle = \frac{1}{2\pi}\sum_{|\nu'|=|\nu|}\sum_{|\tilde\nu'|=|\tilde\nu|}\int_0^\infty \langle u_1, \Psi_{Q+ip,\nu,\tilde\nu}\rangle\langle \Psi_{Q+ip,\nu',\tilde\nu'}, u_2\rangle F^{-1}_{Q+ip}(\nu,\nu')F^{-1}_{Q+ip}(\tilde\nu,\tilde\nu')\,dp}$$

where $\nu, \nu', \tilde\nu, \tilde\nu'$ sum is over Young diagrams and F_{Q+ip} are called Schapovalov matrices (Gram matrices of change of basis).

$$\langle\prod_{i=1}^4 V_{\alpha_i}(x_i)\rangle_{\mathbb{S}^2} = \langle A_{M_1}, A_{M_2}\rangle_{\mathcal{H}}$$

$$= \frac{1}{2\pi}\sum_{|\nu'|=|\nu|}\sum_{|\tilde\nu'|=|\tilde\nu|}\int_0^\infty \langle A_{M_1}, \Psi_{Q+ip,\nu,\tilde\nu}\rangle\langle \Psi_{Q+ip,\nu',\tilde\nu'}, A_{M_2}\rangle F^{-1}_{Q+ip}(\nu,\nu')F^{-1}_{Q+ip}(\tilde\nu,\tilde\nu')\,dp$$

Ward identities and final formula for 4pt function on $\mathbb{S}^2$

For $M_1 = \mathbb{D}$ the disk with two insertions $V_{\alpha_1}(0)$, $V_{\alpha_2}(z)$, Ward identity reads:

$$\begin{aligned}\langle A_{M_1}, \Psi_{Q+ip,\nu,\tilde\nu}\rangle &= \langle A_{M_1}, \prod_{\ell,\tilde\ell} L_{-\nu_\ell}\tilde L_{-\tilde\nu_{\tilde\ell}}\Phi_{Q+ip}\rangle \\ &= \langle A_{M_1}, \Psi_{Q+ip}\rangle w_\nu(\alpha_1,\alpha_2,p)w_{\tilde\nu}(\alpha_1,\alpha_2,p)\bar z^{|\nu|}z^{|\tilde\nu|}|z|^{2(\Delta_{Q+ip}-\Delta_{\alpha_1}-\Delta_{\alpha_2})} \\ &= C(\alpha_1,\alpha_2,Q+ip)w_\nu(\alpha_1,\alpha_2,p)w_{\tilde\nu}(\alpha_1,\alpha_2,p)\bar z^{|\nu|}z^{|\tilde\nu|}|z|^{2(\Delta_{Q+ip}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}\end{aligned}$$

with $w_\nu(\alpha_1,\alpha_2,p)$ algebraic coeff, depend only on Virasoro commutations and $\Delta_{\alpha_j}, \Delta_{Q+ip}$.

Theorem (G-Kupiainen-Rhodes-Vargas '20)

The 4-pt function on the sphere is given by

$$\langle \prod_{i=1}^4 V_{\alpha_i}(x_i)\rangle_{\mathbb{S}^2} = \frac{1}{2\pi}\int_0^\infty \overbrace{C(\alpha_1,\alpha_2,Q-ip)}^{\text{3-pt fct}}\overbrace{C(Q+ip,\alpha_3,\alpha_4)}^{\text{3-pt fct}}|z|^{2(\Delta_{Q+ip}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}_{p,\alpha}(z)|^2 dp$$

Conformal blocks and 3pt function

The term $\mathcal{F}_{p,\alpha}(z)$ is called conformal block, it is a series in z

$$\mathcal{F}_{p,\alpha}(z) = \sum_{n=0}^{\infty} W_n(p, \alpha_1, \alpha_2, \alpha_3, \alpha_4) z^n$$

with $W_n(p, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ algebraic coefficients depending only on $\Delta_{\alpha_j}, \Delta_{Q+ip}$ and Virasoro commutation relations (thus $\boldsymbol{c} = 1 + 6Q^2$).

Corollary (G-Kupiainen-Rhodes-Vargas '20)

The conformal block series converges for almost all p.

Conformal Bootstrap for general surfaces

Theorem (G-Kupiainen-Rhodes-Vargas 21': modular bootstrap)

For a closed Riemann surface (M,g) with m marked points $x=(x_1,\dots,x_m)\in M^m$ and weights $\alpha=(\alpha_1,\dots,\alpha_m)\in(0,Q)^m$, the Liouville correlation functions are given by

$$\boxed{\langle V_{\alpha_1}(x_1)\dots V_{\alpha_n}(x_m)\rangle_{M,g}=C_g\int_{\mathbb{R}_+^{3h-3+m}}\rho(P,\alpha)|\mathcal{F}_{P,\alpha}(q)|^2dP}$$

- $\alpha=(\alpha_1,\dots,\alpha_m)$, $\rho(P,\alpha)$ *is a product of* 3-*point correlations functions on* $\mathbb{S}^2$
- $q\mapsto\mathcal{F}_{P,\alpha}(q)$ *are the holomophic conformal blocks,* $q=(q_1,\dots,q_{3h-3+m})$ *plumbing (complex) coordinates on the moduli space* $\mathcal{M}_{h,m}$ *of Riemann surface of genus* $h=\mathrm{genus}(M)$ *with* m *marked points.*
- $C_g>0$ *an explicit constant depending on choice of representative* g *in each conformal class.*

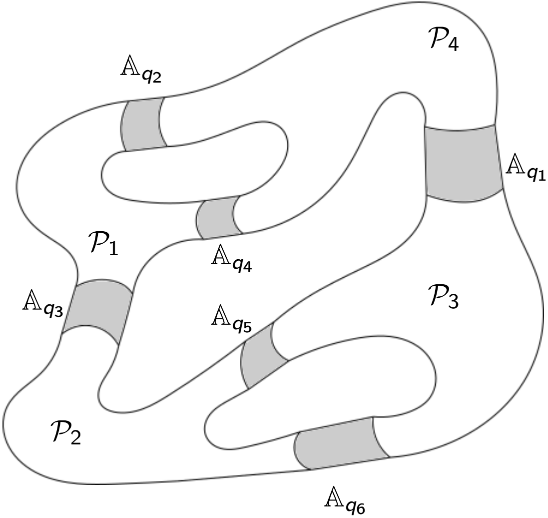


Figure: The plumbed surfaces Σ_{q} with four pairs of pants $\mathcal{P}_1, \dots, \mathcal{P}_4$ and six annuli $\mathbb{A}_{q_1}, \dots, \mathbb{A}_{q_6}$

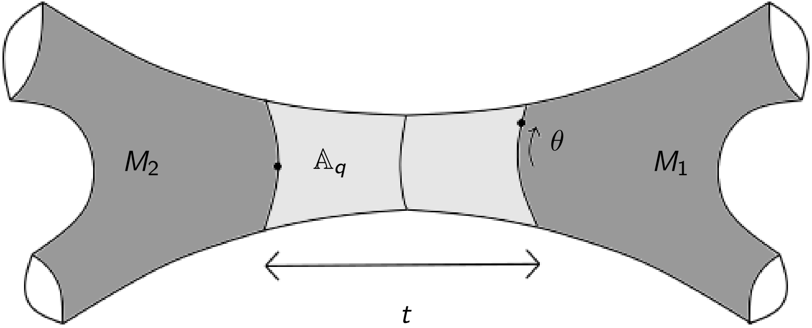
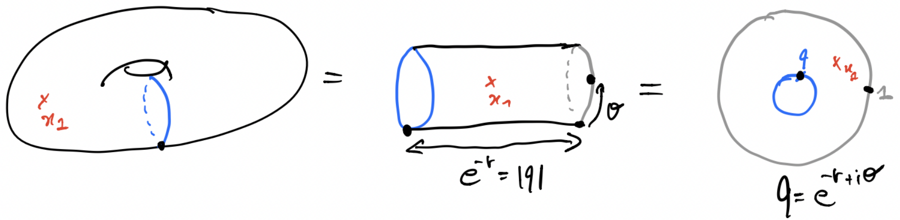


Figure: The plumbing with parameter $q = e^{-t+i\theta}$ of two pairs of pants, viewed as gluing an annulus $\mathbb{A}_q = \{z \in \mathbb{D} \,|\, |q| \leq |z| \leq 1\}$ with a twist of angle θ between the two pairs of pants. The length for the flat metric $|dz|^2/|z|^2$ of the annulus is t.

In terms of amplitudes: composition with $e^{-tH+i\Pi}$ where Π is generator of rotations $z \mapsto e^{i\theta}z$.

Example: torus 1 point



1-point function on torus $\mathbb{T}^2_\tau = \mathbb{C}/(2\pi\mathbb{Z} + 2\pi\tau\mathbb{Z})$, with $q = e^{2i\pi\tau}$

$$\langle V_{\alpha_1}(x_1)\rangle_{\mathbb{T}^2_\tau} = \frac{|q|^{-\frac{1+6Q^2}{12}}}{2\pi}\int_0^\infty C(Q+ip, \alpha_1, Q-ip)|q|^{-2\Delta_{Q+ip}}|\mathcal{F}_{p,\alpha_1}(q)|^2 dp$$

Remarks:

- first mathematical proof of the explicit expressions proposed by physicists (Knizhnik, Belavin, Sonoda, Polchinski, Teschner ...).
- the bootstrap formula depends on the chosen decomposition into pairs of pants, annuli with 1 marked point/insertion and disks with 1 or 2 marked points/insertions
- proves crossing symmetries: formulas for correlations functions given by bootstrap approach do not depend on the decomposition into geometric blocks (although conformal blocks do)
- implies convergence a.e. $P \in \mathbb{R}$ of conformal block series (this was an open problem)

$$\mathcal{F}_{P,\alpha}(q) = \sum_{k \in \mathbb{N}_0^{3h-3+n}} w_k(\alpha, p) q_1^{k_1} \dots q_{3h-3+n}^{k_{3h-3+n}}$$

for $q = (q_1, \dots, q_{3h-3+n}) \in \mathbb{D}^{3h-3+n}$ Marden-Kra plumbing coordinates; here $w_k(\alpha, p)$ are representation theoretic constants depending only on Virasoro commutation relations.

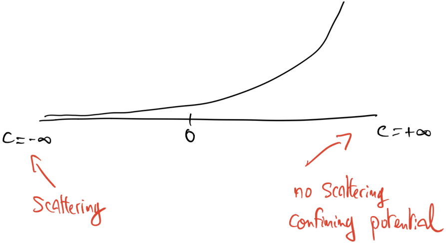
Back to analysis: a (much simpler) toy model

Take the Schrödinger operator

$$\boxed{H_{\mathrm{mod}} = \Delta_{\mathbb{R}\times\mathbb{S}^1} + e^{\gamma c}V = -\partial_c^2 - \partial_\theta^2 + e^{\gamma c}V(\theta)}$$

on a cylinder $\mathbb{R}_c \times \mathbb{S}^1_\theta$, with $V \in L^\infty(\mathbb{S}^1)$ bounded below: $V(\theta) > \varepsilon > 0$.

The potential $e^{\gamma c}$ is confining as $c \to \infty$ and decay at $c \to -\infty$:



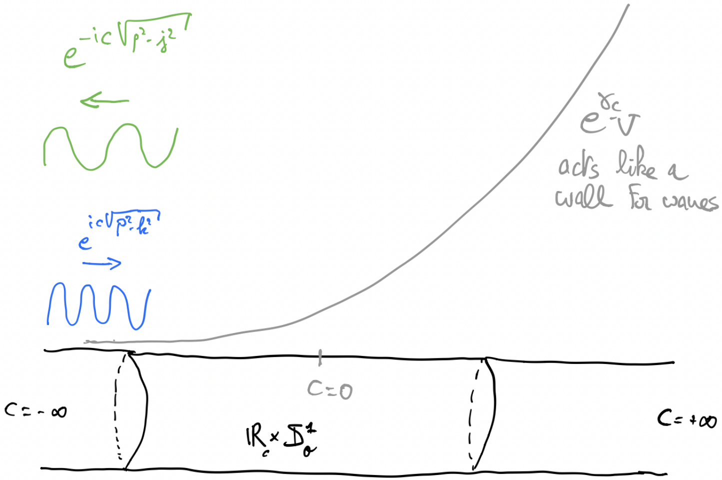
Generalized eigenfunctions diagonalizing $H_{\rm mod}$ have the form (for $p \in \mathbb{R}$, $k \in \mathbb{Z}$, with $p^2 > k^2$)

$$\Phi_{p,k}(c,\theta)|_{\mathbb{R}_-\times\mathbb{S}^1} = e^{ic\sqrt{p^2-k^2}}e^{ik\theta} + \sum_{j\leq k} C_{jk}(p)e^{-ic\sqrt{p^2-j^2}}e^{ij\theta} + r(c,\theta)$$

$$\Phi_{p,k}(c,\theta)|_{\mathbb{R}_+\times\mathbb{S}^1} \in L^2(\mathbb{R}_+\times\mathbb{S}^1)$$

for $r \in e^{-\gamma|c|}L^2$.

- They solve $(H_{\rm mod} - p^2)\Phi_{p,k} = 0$
- $u = e^{ic\sqrt{p^2-k^2}}e^{ik\theta}$ is solution for the free operator $(-\partial_c^2 - \partial_\theta^2 - p^2)u = 0$
- $e^{ic\sqrt{p^2-k^2}}e^{ik\theta}$ is an incoming plane wave, $e^{-ic\sqrt{p^2-j^2}}e^{ij\theta}$ are outgoing plane waves
- $C_{jk}(p)$ = scattering/reflection coefficients



e^{-ic√(p²-j²)}
e^{ic√(p²-k²)}
e^{γc} V
acts like a wall for waves
c = -∞
c = 0
ℝ_c × 𝕊¹_θ
c = +∞

Tool for achieving this:

- prove that the resolvent defined for $\mathrm{Im}(p) > 0$

$$R(p) := (H_{\rm mod} + p^2)^{-1} : L^2 \to L^2$$

admits an analytic continuation down to $\mathrm{Im}(p) = 0$ as operator $e^{-\varepsilon|c|}L^2 \to e^{\varepsilon|c|}L^2$.

- Set $\Phi_{p,k}(c,\theta) = e^{ipc}e^{ik\theta}1_{\mathbb{R}_-}(c) - R(p)(H_{\rm mod} - p^2)(e^{ipc}e^{ik\theta}1_{\mathbb{R}_-}(c))$.

This is proved using parametrix $R_0(p)$

$$(H_{\rm mod} + p^2)R_0(p) = \mathrm{Id} + K(p)$$

with $K(p)$ a compact operator on $e^{-\varepsilon|c|}L^2$, then use Fredholm theorem to invert $\mathrm{Id} + K(p)$.

Parametrix is $R_0(p) = R_0^+(p) + R_0^-(p)$ where

- $R_0^+(p)$ is the resolvent of $H_{\mathrm{mod},\mathbb{R}_+} = H_{\mathrm{mod}}$ acting on $c \in [0, \infty)$ with Dirichlet condition at $c = 0$
$$R_0^+(p) = (H_{\mathrm{mod},\mathbb{R}_+} - p^2)^{-1}$$
well-defined in $\mathrm{Im}(p) \geq 0$ on L^2 outside discrete set on $i\mathbb{R}$ since this operator has discrete spectrum (confining potential).
- $R_0^-(p)$ is the resolvent of $H^0_{\mathrm{mod},\mathbb{R}_-} := -\partial_c^2 - \partial_\theta^2$ acting on $c \in (-\infty, 0]$ with Dirichlet condition at $c = 0$
$$R_0^-(p) = (H^0_{\mathrm{mod},\mathbb{R}_-} - p^2)^{-1}$$
well-defined and explicit (using Fourier analysis), analytic up to $\mathrm{Im} \geq 0$ on weighted spaces $e^{-\varepsilon|c|}L^2 \to e^{\varepsilon|c|}L^2$ (continuous spectrum).

Difficulties in the Liouville CFT case

- $V \in L^{2/\gamma^2-\varepsilon}$ is not bounded and is not bounded below by a positive constant.
- V is a measure when $\gamma > \sqrt{2}$
- $L^2(\mathbb{S}^1)$ is replaced by the Fock space $L^2(\Omega)$ (Ω has infinite dimension)
- the eigenfunctions h_{kl} of P are not bounded and their L^p norms for $p < \infty$ are not bounded uniformly in k, l
- even self-adjointness (domain issues) is not easy

To overcome this, need to use a combination of

- Probabilistic estimates (Feynmann-Kac representation of propagator $e^{-tH} = S(t)$)
- finite negative moments of GMC somehow replace $V > \varepsilon > 0$
- cutoff in frequency k, l and positivity of the operator V
- many tricky arguments.