Integrable systems in geometry and mathematical physics,
in memory of Boris Dubrovin

Geometry and topological recursion

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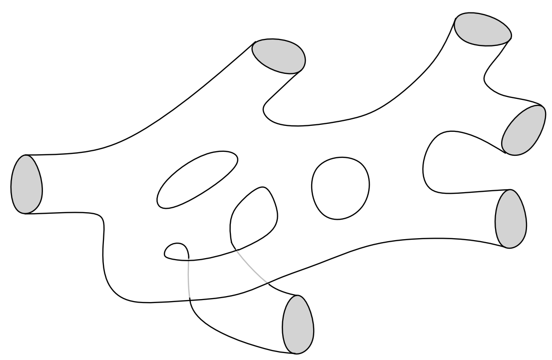
1 Topological recursion: recent history

- Ideas of 15 years ago
- Tools of today

In enumerative geometry of curves / string theory / random 2d geometry …

Compact oriented surface
genus g, n punctures/boundaries



→ **quantities $F_{g,n}$ or $\Omega_{g,n}$**

numeric	*geometric*
$\mathcal{V}^{\otimes n}$	$H^{\bullet}(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{V}^{\otimes n}$ $\mathrm{Fun}(\mathcal{M}_{g,n}, \mathcal{V}^{\otimes n})$

$\xleftarrow{\text{integration}}$

We want to compute $F_{g,n}$ and better understand the *algebraic structures* governing these computations, and their *ubiquity*. E.g.

mirror symmetry $\quad F_{g,n} = \text{periods on } X^n, \quad X = \text{algebraic variety}$

$$\left.\begin{array}{r}\text{non-linear integrable PDEs}\\ \\ \text{linear PDEs}\end{array}\right\} \quad \text{for} \quad Z_{\hbar} = \exp\left(\sum_{g,n} \frac{\hbar^{g-1}}{n!} F_{g,n}\right) \in \mathrm{Fun}_{\hbar}(\mathcal{V})$$

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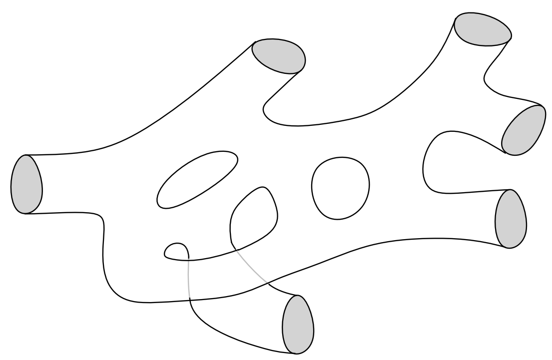

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mirror symmetry $\quad F_{g,n}$ = periods on X^n, $\;X$ = algebraic variety

non-linear integrable PDEs

linear PDEs $\longleftrightarrow$ recursion on $|\chi_{g,n}| = 2g - 2 + n$

for $\quad Z_\hbar = \exp\Big(\sum_{g,n} \frac{\hbar^{g-1}}{n!} F_{g,n}\Big) \in \mathrm{Fun}_\hbar(\mathcal{V})$

universal idea : cutting surfaces into smaller pieces / pasting

1. Chekhov, Eynard, Orantin identified a universal recursive structure governing the formal and asymptotic expansions of matrix integrals

$$d\mu(M) = \frac{1}{Z_N} dM\, e^{-N \mathrm{Tr}\, V(M)}$$

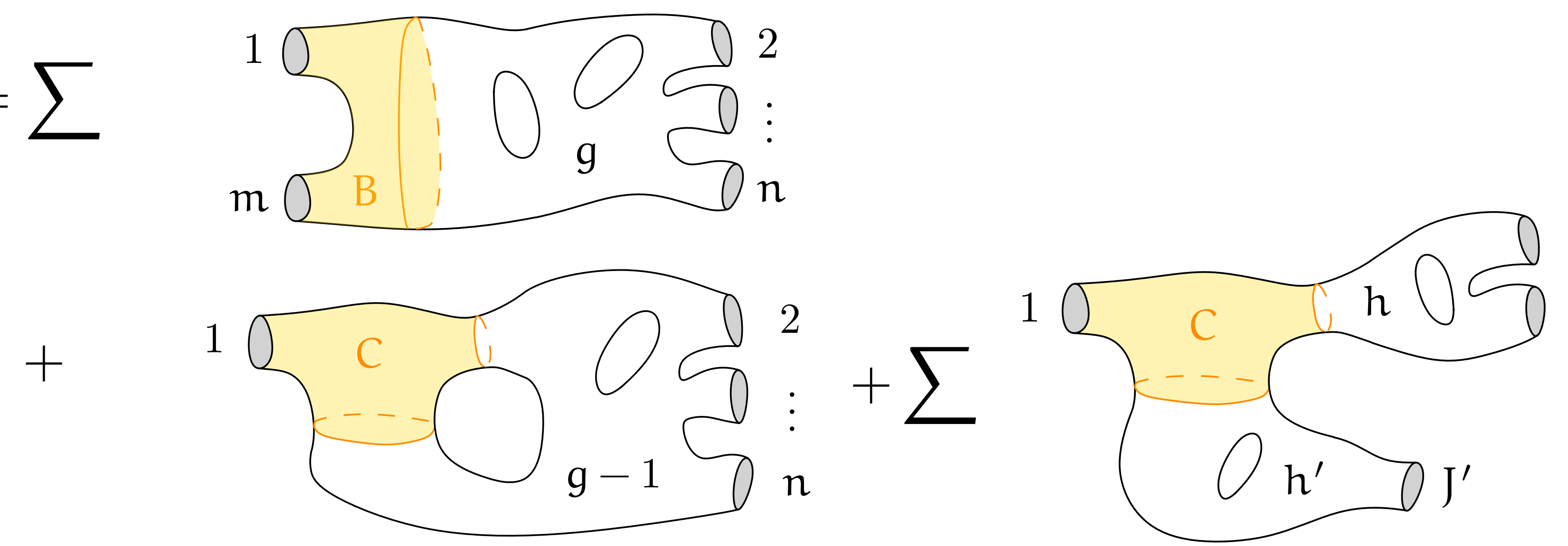
$$\Big\langle \mathrm{Tr}\, \frac{1}{x_1 - M} \cdots \mathrm{Tr}\, \frac{1}{x_n - M} \Big\rangle_c \approx \sum_{g \geqslant 0} N^{2-2g-n} F_{g,n}(x_1, \ldots, x_n)$$

- recursion on 2g - 2 + n
- period computations on the spectral curve $P(x, F_{0,1}) = 0$

They called it **topological recursion (TR)**

$$\text{Terms} \longleftrightarrow \left\{ \begin{array}{l} \text{embeddings of pairs of pants } P \hookrightarrow \Sigma_{g,n} \\ \text{such that } \partial_1 P = \partial_1 \Sigma_{g,n} \text{ and } \Sigma_{g,n} - P \text{ is stable} \end{array} \right\} \Big/ \mathrm{Diff}^{\partial}(\Sigma_{g,n})$$

$F_{g,n} = \sum$



2. **Mirzakhani theorem (07) :** recursion for Weil-Petersson volumes of $\mathcal{M}_{g,n}(L)$

(TR on $y = \dfrac{\sin(\pi\sqrt{2x})}{\pi\sqrt{2x}}$)

… > 10 years ago

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3. Witten conjecture/Kontsevich + Dijkgraaf-Verlinde-Verlinde theorem (91)

Virasoro constraints for $F_{g,n} = \sum_{k_1,\dots,k_n \geqslant 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \right) \prod_{i=1}^{n} \dfrac{(2k_i+1)!!dy_i}{y_i^{2k_i+2}}$

(TR on $x = \dfrac{y^2}{2}$)

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(TR on $x = \frac{y^2}{2}$)

4. **Remodeling the B-model (conjecture of Bouchard, Klemm, Mariño, Pasquetti, 07)**

TR on mirror curve $P(e^x, e^y) = 0$ of a toric CY3 X
computes its open Gromov-Witten theory

$\Rightarrow$ **Bouchard-Mariño conjecture (08)**

TR on $e^x = ye^{-y}$ computes simple Hurwitz numbers

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5. Norbury-Scott conjecture : TR on $x = e^y + e^{-y}$
computes the (stationary) Gromov-Witten theory of $\mathbb{CP}^1$

Why/How does TR appears in a given problem ?

I. Via analysis of functional relations

Schwinger-Dyson equations / Tutte's recursion on Feynman graphs (matrix models)
1,3, non-rigorous derivations for 4,5

Cut-and-join equations in Hurwitz theory
Bouchard-Mariño conjecture proved by Eynard, Mulase, Safnuk (11)
More general Hurwitz theory ???

II. Via geometry

Isolated example of Mirzakhani : recursive partition of unity
on Teichmüller space, whose integration yields TR for volumes

III. Relation to Frobenius manifolds & Givental formalism in GW-theory ???

Nowadays: many ways to prove TR

I. Via analysis of Schwinger-Dyson equations

TR for large class of matrix models (multitrace, multicut) B, Eynard, Orantin 13, B. 15

Existence of asymptotic expansions properly justified Albeverio, Pastur, Shcherbina (01) B, Guionnet, Kozlowski (11-15)

I'. Via analysis of cut-and-join equations/semi-infinite wedge formulas

Amsterdam/Moscow school, Bouchard, Mulase, Norbury, Lewanski, Do, Karev, B., Moskowsky (2011-2021)
Alexandrov, Chapuy, Eynard, Harnad

I''. Reconstruction of formal WKB expansions

Bergere, Eynard, B, Iwaki, Marchal, Dumitrescu, Mulase, Orantin, Garcia-Failde (2009-…)

II. Via geometric recursions Andersen, B, Orantin (17-…)

III. $\left\{\right.$ **Relation to Frobenius manifolds & Givental formalism & CohFTs**
Dunin-Barkowski, Orantin, Spitz, Shadrin, Norbury, Popolitov (12-16)

Intersection theory on $\overline{\mathcal{M}}_{g,n}$
Eynard (12) + …

IV. Representation theory of VOAs

Orantin, Kostov (2010) Milanov (2015), B., Bouchard, Chidambaram, Creutzig, Noshchenko (2017-…)

Nowadays: many ways to prove TR

I. Via analysis of Schwinger-Dyson equations

TR for large N expansion of SU(N) Chern-Simons theory

I′. Via analysis of cut-and-join equations/semi-infinite wedge formulas

For all weighted double Hurwitz numbers and spin Hurwitz numbers

I″. Reconstruction of formal WKB expansions

TR for Painleve tau-functions, for large class of Hurwitz problems, …

II. Via geometric recursions

III.

- **Relation to Frobenius manifolds/Givental formalism/CohFTs**
  Proof of Norbury-Scott conjecture
- **Intersection theory on $\overline{\mathcal{M}}_{g,n}$**
  Proof of remodeling B-model conjecture

IV. Representation theory of VOAs

Nowadays: TR can emerge in many ways

I. Via analysis of Schwinger-Dyson equations

TR for large N expansion of SU(N) Chern-Simons theory

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  Proof of Norbury-Scott conjecture, of the remodeling B-model conjecture
- **Intersection theory on** $\overline{\mathcal{M}}_{g,n}$

IV. Representation theory of VOAs

2 Geometric recursions

- Mirzakhani identity
- Generalisations and applications

1. Mirzakhani identity

- Σ compact oriented smooth surface, genus g, n boundaries

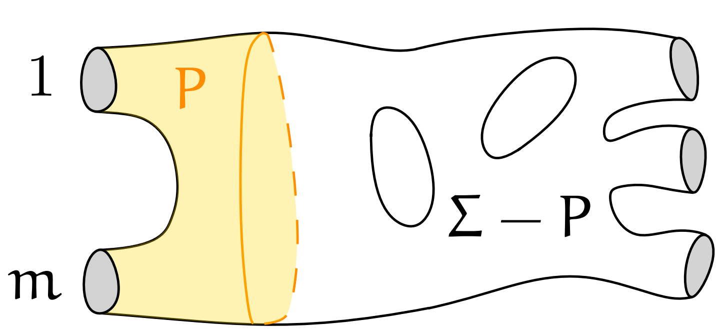
- $\mathcal{T}_\Sigma$ = Teichmüller space = $\left\{ \begin{array}{l} \text{hyperbolic metrics on } \Sigma \\ \text{with geodesic boundaries} \end{array} \right\} \Big/ \mathrm{Diff}_0(\Sigma, \partial\Sigma)$

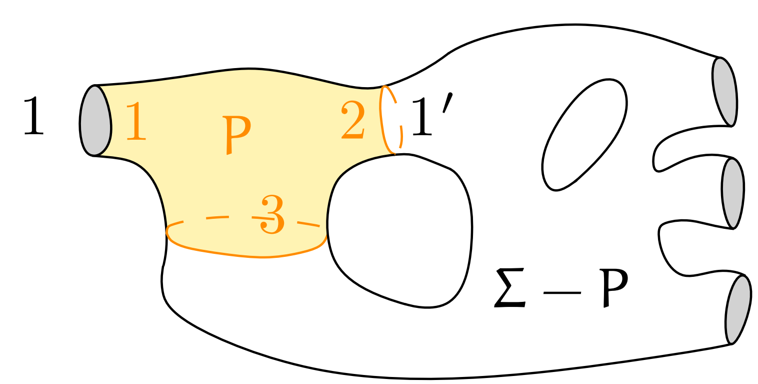
- $\mathcal{M}_\Sigma(L) = \mathcal{T}_\Sigma(L)/\Gamma_\Sigma^\partial$ = moduli space of bordered Riemann surfaces with fixed boundary lengths $L \in \mathbb{R}_+^n$

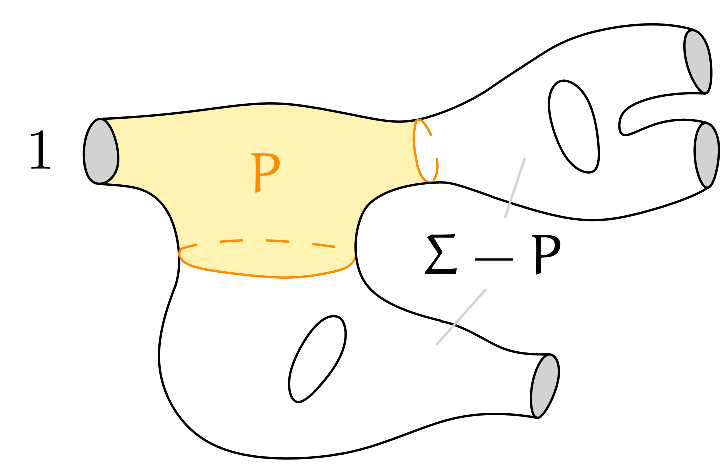
equipped with μ_{WP} = Weil-Petersson (symplectic) volume form

- $\mathcal{P}_\Sigma = \left\{ \begin{array}{l} \text{isotopy class of } P \hookrightarrow \Sigma \text{ with labeled boundaries} \\ \text{such that } \partial_1 P = \partial_1 \Sigma \text{ and } \Sigma - P \text{ is stable} \end{array} \right\}$

$$= \left(\bigsqcup_{m=2}^{n} \mathcal{P}_\Sigma^m \right) \sqcup \mathcal{P}_\Sigma^\emptyset$$







1. Mirzakhani identity

$$B_{\mathrm{M}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big(F(L_1 + L_2 - \ell) + F(L_1 - L_2 - \ell) - F(-L_1 + L_2 - \ell) - F(-L_1 - L_2 - \ell)\big)$$

$$C_{\mathrm{M}}(L_1, \ell, \ell') = \frac{1}{L_1}\big(F(L_1 - \ell - \ell') - F(-L_1 - \ell - \ell')\big) \qquad \text{with } F(x) = 2\,\ln(1 + e^{x/2})$$

Theorem (Mirzakhani, 07) For $2g - 2 + n \geq 2$ and any $\sigma \in \mathcal{T}_\Sigma$

(a)

$$1 = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^m} B_{\mathrm{M}}(\vec{\ell}_\sigma(\partial P)) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_\Sigma^\emptyset} C_{\mathrm{M}}(\vec{\ell}_\sigma(\partial P))$$

(b) Topological recursion for the WP volumes

Terms $\longleftrightarrow$ $\mathcal{P}_\Sigma/\mathrm{Diff}_\Sigma^\partial$ (finite)

$$V_{g,n}(L) = \sum_{m=2}^{n} \int_{\mathbb{R}_+} d\ell\, \ell\, B_{\mathrm{M}}(L_1, L_m, \ell)\, V_{g,n-1}(\ell, L \setminus \{L_m\})$$

$$+ \frac{1}{2} \int_{\mathbb{R}_+^2} d\ell d\ell'\, \ell\ell' C_{\mathrm{M}}(L_1, \ell, \ell') \Bigg(V_{g-1,n+1}(\ell, \ell', L) + \sum_{\substack{J \sqcup J' = L \setminus \{L_1\} \\ h + h' = g}} V_{h,1+|J|}(\ell, J) V_{h',1+|J'|}(\ell', J') \Bigg)$$

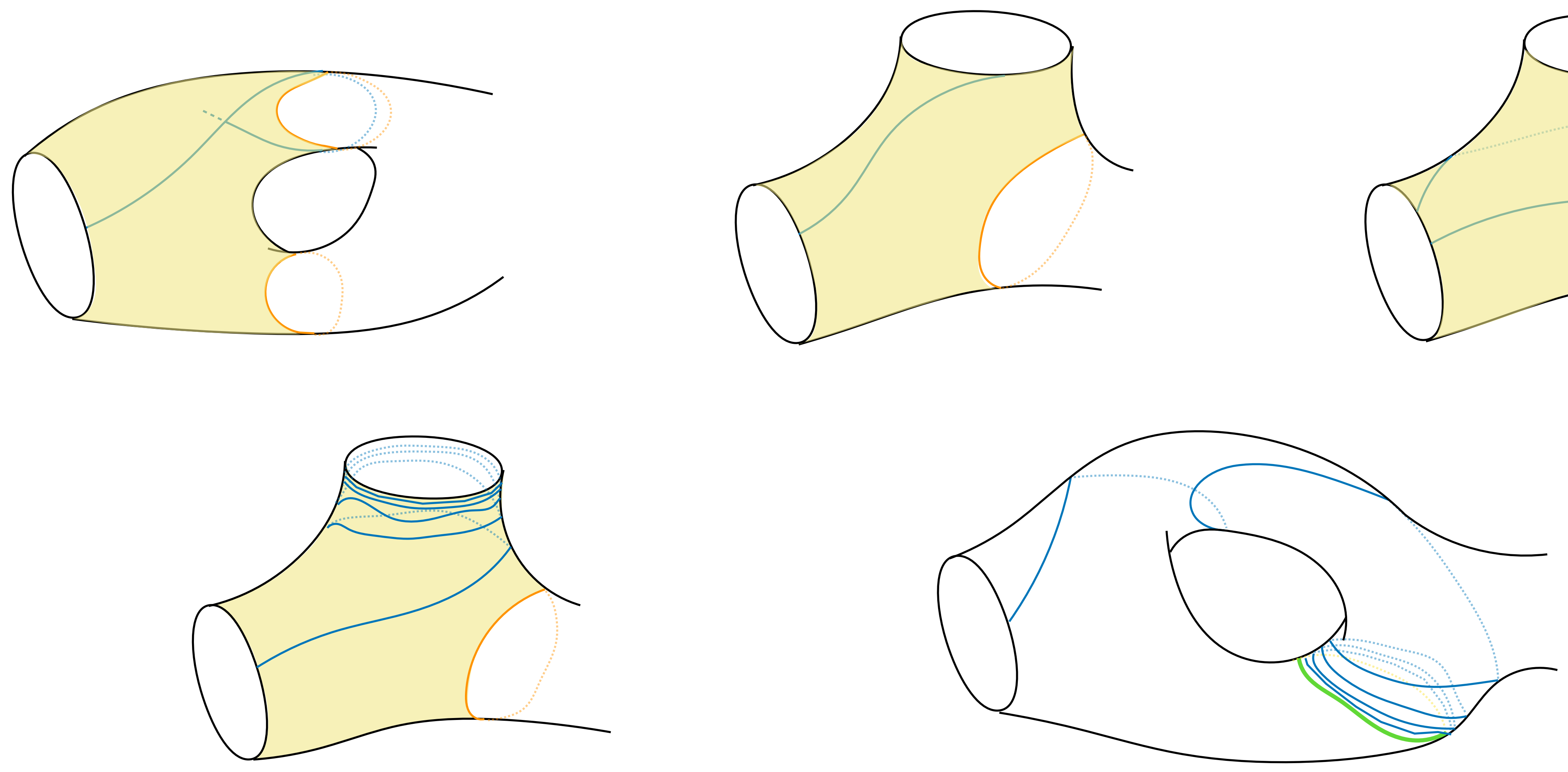
1. Mirzakhani identity

Idea of the proof Let $\sigma \in \mathcal{T}_\Sigma$

$x \in \partial_1\Sigma \rightsquigarrow \gamma_x$ geodesic issuing from $x \perp \partial_1\Sigma$, stopped at first intersection point

$\rightsquigarrow [P_x] \in \mathcal{P}^1_\Sigma$ determined by tubular neighboorhood of $\partial_1\Sigma \cup \gamma_x$

when the geodesic does not accumulate on $\alpha \subset \mathring{\Sigma}$



1. Mirzakhani identity

Idea of the proof (continued)

We have an almost everywhere defined map $\partial_1\Sigma \dashrightarrow \mathcal{P}_{\Sigma}$

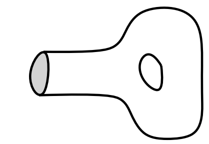
$$1 = \frac{1}{\ell_\sigma(\partial_1\Sigma)} \sum_{[P]\in\mathcal{P}_{\Sigma}} \ell_\sigma\big(\{x \in \partial_1\Sigma \quad | \quad [P_x] = [P]\}\big)$$

Given $[P]$, one can identify the set of points $x \in \partial_1\Sigma$ intrinsically and compute their measure by hyperbolic trigonometry

Key for integration is that Fenchel-Nielsen coordinates are

- compatible with cutting/gluing
- canonical for Weil-Petersson symplectic form (Wolpert formula)

2. TR from geometric recursion

- Initial data $A, B, C \in C^0(\mathcal{T}_P) \cong C^0(\mathbb{R}^3_+)$ and $D \in C^0(\mathcal{T}_T)$ $T =$ 

  $X(L_1, L_2, L_3) = X(L_1, L_3, L_2)$ for $X = A, C$

- $|\chi| = 1$ $\quad \Omega_P = A \qquad \Omega_T = D$

  union $\quad \Omega_{\Sigma_1 \sqcup \Sigma_2}(\sigma_1, \sigma_2) = \Omega_{\Sigma_1}(\sigma_1)\Omega_{\Sigma_2}(\sigma_2)$

  $|\chi| \geqslant 2$ $$\Omega_\Sigma(\sigma) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}^{b}_{\Sigma}} B(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma - P}(\sigma|_{\Sigma - P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}^{\emptyset}_{\Sigma}} C(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma - P}(\sigma|_{\Sigma - P})$$

Theorem Andersen B Orantin 17

If A,B,C,D satisfy some (explicit) bounds

$\Sigma \longmapsto \Omega_\Sigma \in C^0(\mathcal{T}_\Sigma)$ is a well-defined and invariant under $\Gamma^{\partial}_{\Sigma}$
(absolute convergence on any compact)

2. TR from geometric recursion

GR formula $$\Omega_\Sigma(\sigma) = \sum_{b=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^b} B(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma-P}(\sigma|_{\Sigma-P}) + \tfrac{1}{2}\sum_{[P]\in\mathcal{P}_\Sigma^\emptyset} C(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma-P}(\sigma|_{\Sigma-P})$$

Theorem Andersen B Orantin 17

Under assumptions, $\Sigma \longmapsto \Omega_\Sigma \in C^0(\mathcal{T}_\Sigma)$ is well-defined, Γ_Σ^∂-invariant

and $$F_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \Omega_{\Sigma_{g,n}} d\mu_{WP}(\sigma)$$ satisfies TR

$$F_{g,n}(L) = \sum_{m=2}^{n} \int_{\mathbf{R}_+} d\ell\,\ell\,B(L_1,L_m,\ell)\,F_{g,n-1}(\ell, L\setminus\{L_m\})$$
$$+\frac{1}{2}\int_{\mathbf{R}_+^2} d\ell d\ell'\,\ell\ell' C(L_1,\ell,\ell')\Big(F_{g-1,n+1}(\ell,\ell',L) + \sum_{\substack{J\sqcup J'=L\setminus\{L_1\}\\ h+h'=g}} F_{h,1+|J|}(\ell,J)F_{h',1+|J'|}(\ell',J')\Big)$$

Key for integration is that Fenchel-Nielsen coordinates are

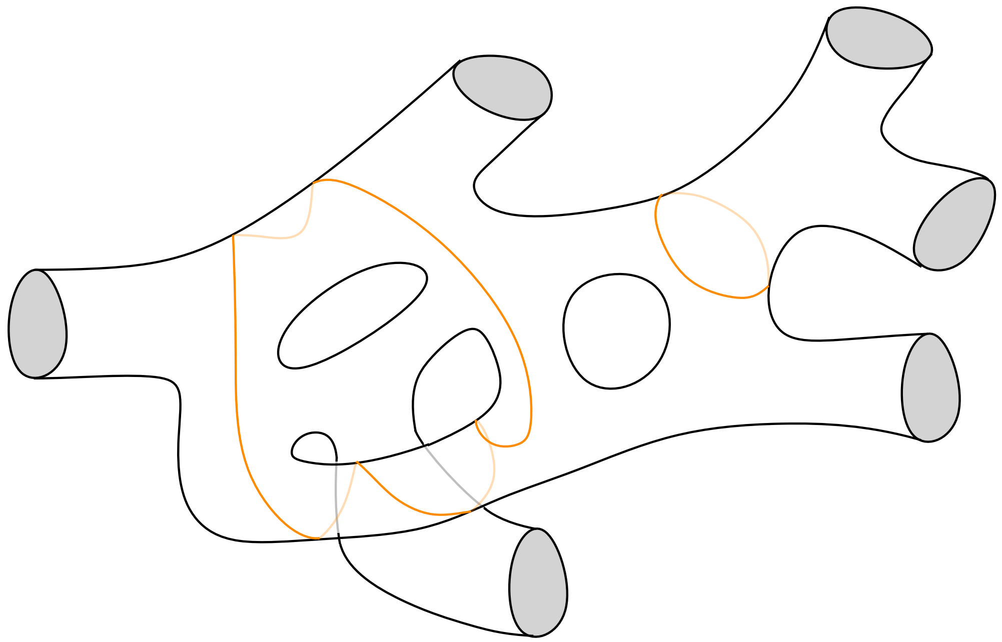
- compatible with cutting/gluing
- canonical for Weil-Petersson symplectic form

Sees TR as a shadow (after integration over moduli) of finer geometric recursions

3. Applications

Generalization of Mirzakhani identities

$$M'_{\Sigma} = \left\{ \text{primitive multicurves on } \ \Sigma \right\}$$



Theorem Andersen B Orantin 18

For any test function $f \in C^0(\mathbb{R}_+)$ with fast decay

$$\Omega^M[f](\sigma) = \sum_{\gamma \in M'_{\Sigma}} \prod_{c \in \pi_0(\gamma)} f(\ell_\sigma(c))$$ is computed by GR for twisted initial data

$$A^M[f](L_1, L_2, L_3) = A^M(L_1, L_2, L_3)$$

$$B^M[f](L_1, L_2, \ell) = B^M(L_1, L_2, \ell) + f(\ell) A^M(L_1, L_2, \ell)$$

$$C^M[f](L_1, \ell, \ell') = C^M(L_1, \ell, \ell') + f(\ell) B^M(L_1, \ell, \ell') + f(\ell') B^M(L_1, \ell', \ell) + f(\ell) f(\ell') A^M(L_1, \ell, \ell')$$

$$D^M[f](\sigma) = \sum_{\gamma \text{ simple}} C^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma)) + f(\ell_\sigma(\gamma)) A^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma))$$

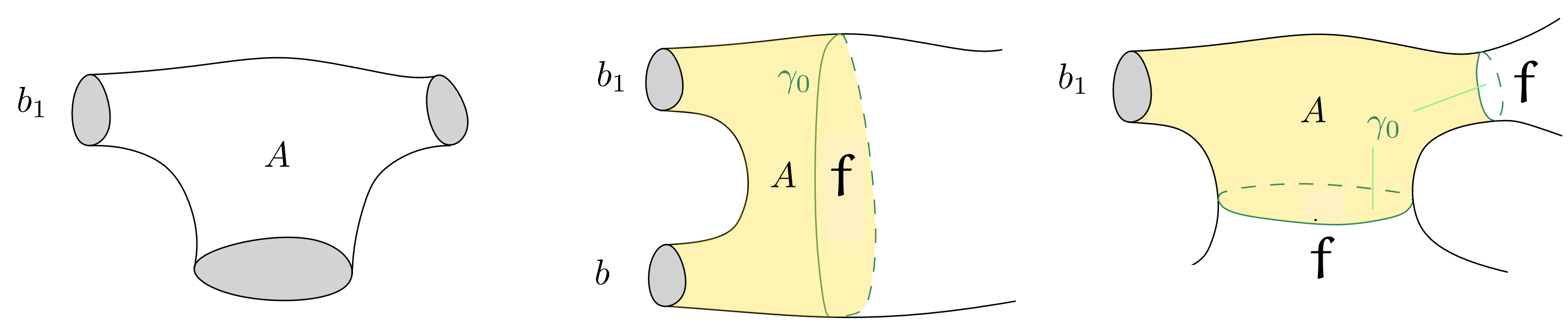
3. Applications

Idea of the proof same in hyperbolic or combinatorial setting

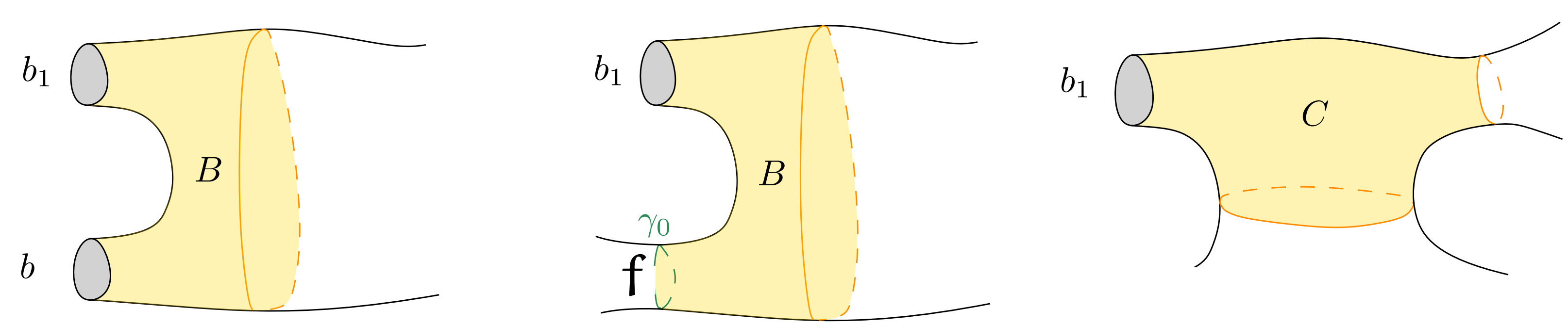
$$\begin{aligned}
\Omega^{M}[f](\sigma) &= \sum_{\gamma \in M'_{\Sigma}} \prod_{c \in \pi_0(\gamma)} f(\ell_\sigma(c)) \cdot \mathbf{1}_{\Sigma - \gamma}(\sigma|_{\Sigma-\gamma}) \\
&= \sum_{\gamma \in M'_{\Sigma}} \prod_{c \in \pi_0(\gamma)} f(\ell_\sigma(c)) \sum_{[P] \in \mathcal{P}_{\Sigma_{-\gamma}}} X_P^{M}(\sigma|_{\Sigma - P}) \\
&= \sum_{[P] \in \mathcal{P}_{\Sigma}} \sum_{\gamma \in M'_{\Sigma - P}} \cdots
\end{aligned}$$

use Mirzakhani identity

and collect the weights



$A = \Omega_{0,3} \equiv 1$



3. Applications

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$$B^M[f](L_1, L_2, \ell) = B^M(L_1, L_2, \ell) + f(\ell) A^M(L_1, L_2, \ell)$$

$$C^M[f](L_1, \ell, \ell') = C^M(L_1, \ell, \ell') + f(\ell) B^M(L_1, \ell, \ell') + f(\ell') B^M(L_1, \ell', \ell) + f(\ell) f(\ell') A^M(L_1, \ell, \ell')$$

$$D^M[f](\sigma) = \sum\nolimits_{\gamma \text{ simple}} C^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma)) + f(\ell_\sigma(\gamma)) A^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma))$$

Consequences

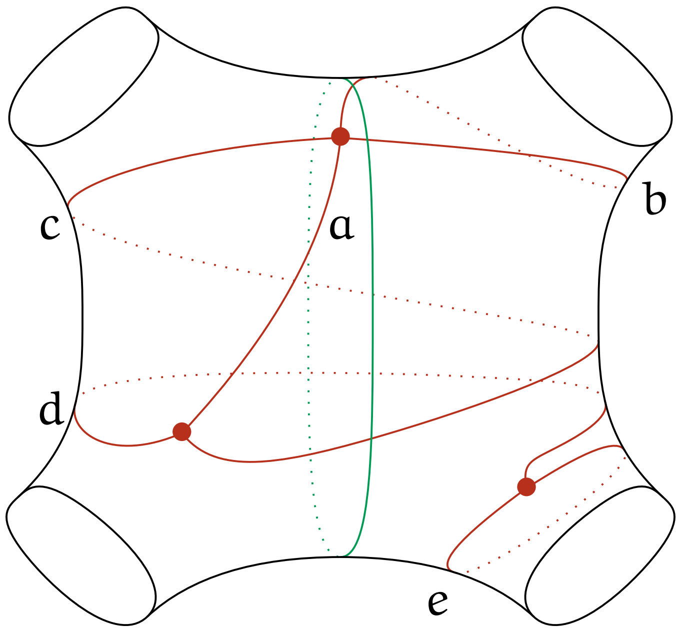
TR computes WP-averages of multicurve statistics

TR computes Masur-Veech volumes of moduli of quadratic differentials

Andersen, B., Charbonnier, Delecroix, Giacchetto, Lewanski, Wheeler 19

4. Combinatorial geometry

$$\mathcal{T}^{\mathrm{comb}}_{\Sigma} = \left\{ \begin{array}{l} \text{embedded metric ribbon graphs } \mathbb{G} \hookrightarrow \Sigma \\ \text{s.t. } \Sigma \text{ retracts onto } \mathbb{G}, \text{ up to isotopy} \end{array} \right\}$$



is homeomorphic to $\mathcal{T}_{\Sigma}$

but a different (symplectic) geometry than WP geometry

Kontsevich 2-form on $\mathcal{T}^{\mathrm{comb}}_{\Sigma}(L)$

$\Gamma^{\partial}_{\Sigma}$ - invariant

$$\omega_{\mathrm{K}} = \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{e<e' \\ \text{around } \partial_i \Sigma}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{e'}$$

Associated volume form μ_{K}

Combinatorial volumes

$$\int_{\mathcal{M}^{\mathrm{comb}}_{g,n}(L)} \mathrm{d}\mu_{\mathrm{K}} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2} \psi_i \right)$$

Kontsevich 91, Zvonkine 03

4. Combinatorial geometry

$$\mathcal{T}_\Sigma^{\mathrm{comb}} = \left\{ \begin{array}{c} \text{embedded metric ribbon graphs } \mathbb{G} \hookrightarrow \Sigma \\ \Sigma \text{ retracts onto } \mathbb{G} \end{array} \right\} \quad \text{homeo. to} \quad \mathcal{T}_\Sigma$$

but different (symplectic) geometry : Kontsevich μ_K vs. Weil-Petersson μ_{WP}

Kontsevich 91
Zvonkine 03

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} d\mu_K = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2} \psi_i \right)$$

Theorem Andersen B Charbonnier Giacchetto Lewanski Wheeler 21

There are combinatorial FN coordinates $\mathcal{T}_\Sigma^{\mathrm{comb}} \to (\mathbb{R}^+ \times \mathbb{R})^{3g-3+n} \times \mathbb{R}_+^n$

Image is open dense with zero measure complement

$w_K = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i$ on locus with fixed boundary lengths

One can set up geometric recursion to get $\Omega_\Sigma \in C^0(\mathcal{T}_\Sigma^{\mathrm{comb}})$

and $F_{g,n}(L) = \int_{\mathcal{M}_\Sigma^{\mathrm{comb}}(L)} \Omega_\Sigma \, d\mu_K$ satisfy TR

4. Combinatorial geometry

Andersen B Charbonnier Giacchetto Lewanski Wheeler 21

Most tools of hyperbolic geometry have an analogue in $\mathcal{T}_{\Sigma}^{\mathrm{comb}}$

Theorem (combinatorial Mirzakhani-McShane identity)

$$A^{K}(L_1, L_2, L_3) = 1$$

$$B^{K}(L_1, L_2, \ell) = \frac{1}{2L_1}\big([L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+ + [L_1 + L_2 - \ell]_+\big)$$

$$C^{K}(L_1, \ell, \ell') = \frac{1}{L_1}[L_1 - \ell - \ell']_+$$

$$D^{K}(\mathbb{G}) = \sum_{\gamma \text{ simple}} C^{K}\big(\vec{\ell}_{\mathbb{G}}(\partial(\Sigma - \gamma))\big)$$

$$\begin{array}{ccc} \text{(identity above)} & \xrightarrow{\ \mathbf{TR}\ } & \\ \Big\downarrow \mathbf{GR} & & \\ \Omega_{\Sigma}^{K} \equiv 1 & \xrightarrow{\ \text{integration}\ } & F_{g,n}(L) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\big(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\big) \end{array}$$

Fully geometric proof of Virasoro constraints of Witten's conjecture
(bypassing matrix models/integrability)

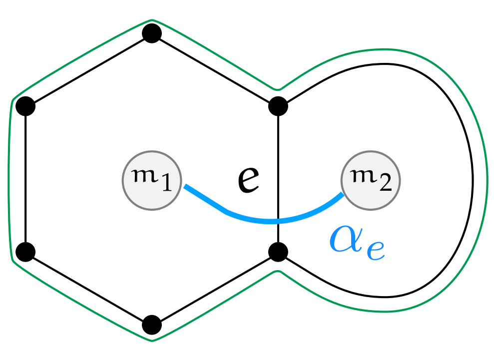
4. Combinatorial geometry

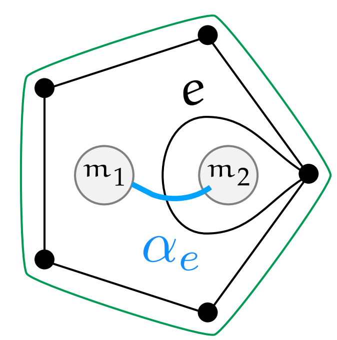
Idea of proof Let $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}$

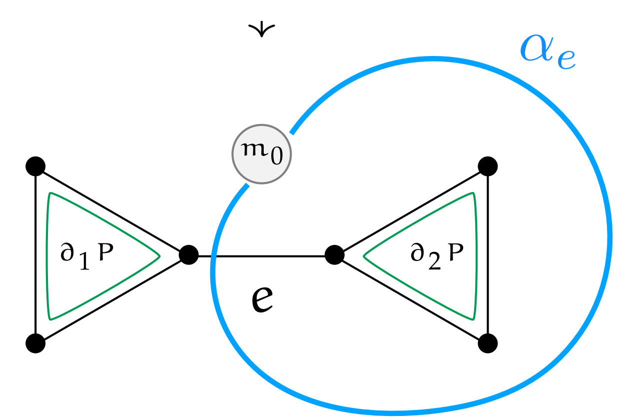
We shoot geodesics, but the starting point is already the ending point (on the graph)

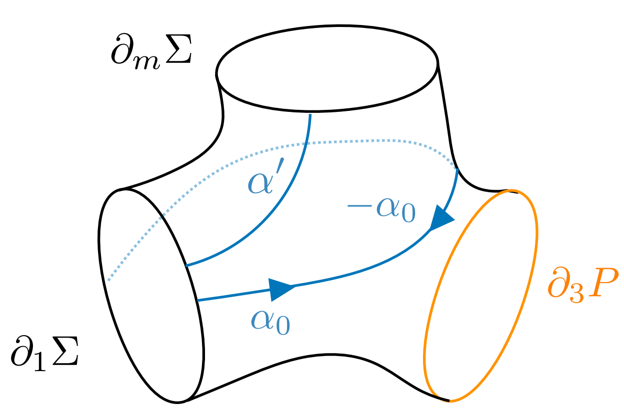
Write $$1 = \frac{1}{\ell_{\mathbb{G}}(\partial_1\Sigma)} \sum_{\substack{e = \text{edge} \\ \text{around } \partial_1\Sigma}} \ell_{\mathbb{G}}(e) = \frac{1}{\ell_{\mathbb{G}}(\partial_1\Sigma)} \sum_{\substack{\alpha \text{ arc} \\ \text{(dual edge)}}} \ell_{\mathbb{G}}(\alpha) = \sum_{[P] \in \mathcal{P}_\Sigma} \sum_{\alpha \in Q^{-1}([P])} \ell_{\mathbb{G}}(\alpha)$$

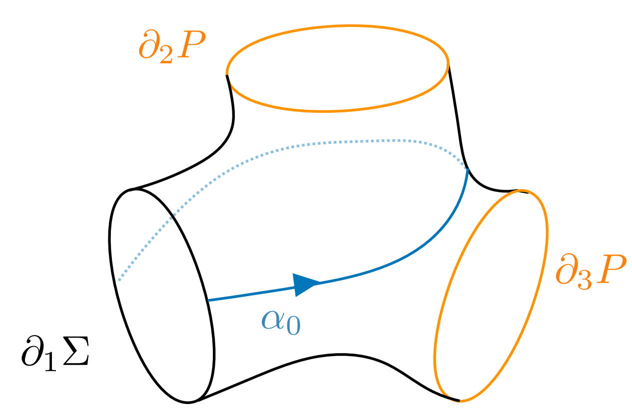
Partition the sum according to the isotopy class of pair of pants $Q(P_\alpha) \in \mathcal{P}_\Sigma$ determined by the tubular neighborhood of $\partial_1\Sigma \cup \alpha$











5. Future directions

- Differential-geometric approach to Witten r-spin conjecture is missing

Construction of Witten r-spin class	Polishchuk, Vaintrob 01, Chiodo 02
Proof of Witten r-spin conjecture (rKdV)	Faber, Shadrin, Zvonkine 06
Proof of W-constraints / TR	Milanov 16

- Analogue for surfaces with corners ? in progress, Andersen, B., Orantin

What can be hoped for :

- open Mirzakhani identity

- twisting: geometric recursion for statistics of length spectrum

- via Selberg trace formula: geometric recursion for statistics of Laplace spectrum

- integration over the open moduli space

3

Representation theory of VOAS

- Airy structures from VOAS
- Supersymmetric gauge theories
- Intersection theory

1. Airy structures

$\mathscr{D}_{\mathcal{V}}^{\hbar} = \mathbb{C}[[\hbar]]\langle x_i, \hbar\partial_{x_i} \quad i \in I\rangle \qquad \deg x_i = 1 \qquad \deg \hbar = 2$

graded algebra of differential operators on $\mathcal{V}$

basis $(e_i)_{i \in I}$

(dual) linear coordinates $(x_i)_{i \in I}$

A **quantum Airy structure** is a linear map $\mathcal{L} : \mathcal{V} \to \mathscr{D}_{\mathcal{V}}^{\hbar}$ such that

- $\mathcal{L}(e_i) = \hbar\partial_{x_i} + O(2)$
- $[\mathcal{L}, \mathcal{L}] \subseteq \hbar\mathscr{D}_{\mathcal{V}}^{\hbar} \cdot \mathcal{L}$

Theorem (Kontsevich, Soibelman 17)

For a given quantum Airy structure, there exists a unique

$$F = \sum_{\substack{2g-2+n>0 \\ n>0}} \frac{\hbar^g}{n!} F_{g,n} \qquad F_{g,n} \in \mathrm{Sym}^n \mathcal{V}^*$$

satisfying $\quad \forall v \in \mathcal{V} \qquad \mathcal{L}(v) \cdot e^{F/\hbar} = 0$

1. Airy structures

Kontsevich Soibelman 17

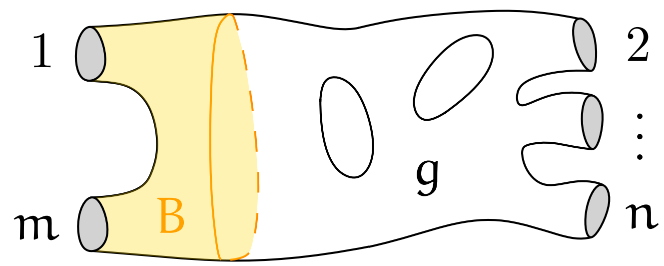
Andersen B Chekhov Orantin 17

initial data (A, B, C, D) $\xrightarrow{\textbf{Topological recursion (TR)}}$ $F_{g,n} \in \mathrm{Sym}^n \mathcal{V}^*$

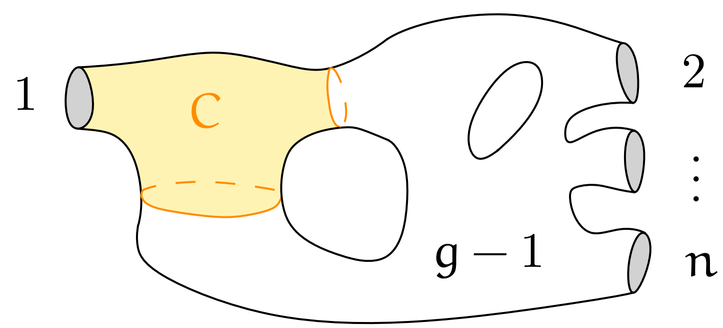
Degree 2 case $\mathcal{L}(e_i) = \hbar\partial_{x_i} - \sum_{a,b} \left(\frac{1}{2}A^i_{a,b}x_a x_b + B^i_{a,b}x_a\,\hbar\partial_{x_b} + \frac{1}{2}C^i_{a,b}\,\hbar^2\partial_{x_a}\partial_{x_b}\right) - \hbar D^i$

$|\chi| = 1$ $F_{0,3} = A$ $F_{1,1} = D$

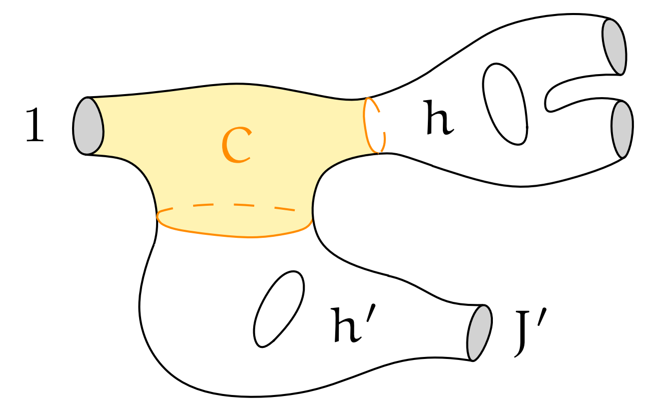
$|\chi| \geqslant 2$ $F_{g,n} = \sum$



$+$



$+\sum$



Identifies the minimal algebraic conditions for the emergence of TR

1. Airy structures

- (Quantum) Airy structures are not easy to find !

- Large abelian group of symmetries (containing Givental group)

  E.g. $\mathcal{U} = \exp(\frac{\hbar}{2} u_{a,b} \partial_{x_a} \partial_{x_b})$

$\mathcal{L}_i \to \mathcal{U}\mathcal{L}_i\mathcal{U}^{-1}$ $\longrightarrow$ $e^{F/\hbar} \to \mathcal{U}e^{F/\hbar}$

$x_i \to x_i + u_{i,a}\,\hbar\partial_{x_a}$

$A[u]^i_{j,k} = A^i_{j,k}$

$B[u]^i_{j,k} = B^i_{j,k} + u_{j,a}A^i_{a,k}$

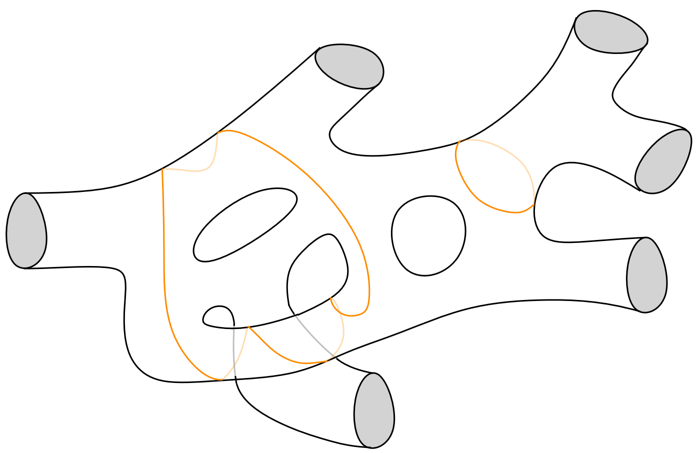
$C[u]^i_{j,k} = C^i_{j,k} + u_{j,a}B^i_{a,k} + u_{k,a}B^i_{a,j} + u_{j,a}u_{k,b}A^i_{a,b}$

$D[u]^i = D^i + \frac{1}{2}\,u_{a,b}A^i_{a,b}$

$F_{g,n}$ $\to$ sum over $M'_\Sigma/\Gamma^\partial_\Sigma$
(stable graphs)

vertex weights $F_{h(v),k(v)}$

edge weights u



Twisting is a geometric lift of this symmetry

2. Construction from VOAs

Vertex operator algebra (2d chiral CFT)
+ twist by an automorphism σ $\quad\rightsquigarrow\quad$ quantum Airy structures
+ representation in $\mathcal{D}^{\hbar}_{\mathcal{V}}$

Fundamental example $\quad \mathcal{W}(\mathfrak{gl}_r)_{\mathfrak{c}} \qquad$ (= Virasoro for $r = 2$)

generators $H^{\alpha}_{k} \quad k \in \mathbb{Z} \quad \alpha \in \{1, \ldots, r\}$

automorphisms $\quad \mathfrak{c} = r \quad \mathfrak{S}_r \ltimes \mathbb{Z}_2$

$\mathfrak{c} \neq r \quad \mathbb{Z}_2$

admit representations in $\mathcal{D}^{\hbar}_{\mathbb{C}^r[[z]]}$
(quantum Miura transform)

With help of representation theory, one can find subsets of modes $\mathcal{H}$

satisfying the (tricky) ideal condition $[\mathcal{H}, \mathcal{H}] \subseteq \hbar \mathcal{W}(\mathfrak{gl}_r) \cdot \mathcal{H}$

By suitable translations $x_k \rightarrow x_k + t_k$, one can get the degree condition

There is a tension between choice of ideal vs. choice of translation

2. Construction from VOAs

Fundamental example $\mathcal{W}(\mathfrak{gl}_r)_{\mathfrak{c}}$

generators H^i_k $k \in \mathbb{Z}$ $i \in \{1, \ldots, r\}$ have representation in $\mathcal{D}^{\hbar}_{\mathbb{C}^r[[z]]}$

Theorem B Bouchard Chidambaram Creutzig Noshchenko 18

For $\mathfrak{c} = r$, twist by $\sigma = (1 \cdots r)$, $s \in \{1, \ldots, r+1\}$ such that $r = \pm 1 \bmod s$

$\mathcal{H} = \{ H^i_k : \; rk + s(i-1) + \delta_{i,1} > 0\}$ with translation $x_s \to x_s - 1/s$

⟶ free energies $F^{(r,s)}_{g,n}$ computed by TR

Likewise with $\sigma = (1\,2 \cdots r-1) \in \mathfrak{S}_r$ and $s|r$

⟶ free energies $\widetilde{F}^{(r,s)}_{g,n}$ computed by TR

General twist $\sigma \in \mathfrak{S}_r$: many more (almost classified)
B Kramer Schüler 20

Untwisted cases ?

3. Spectral curve description

A **spectral curve** is a branched cover $\tilde{C} \xrightarrow{x} \mathbb{C}$ together with

$\mathfrak{a} = $ zeroes of dx

$y : \tilde{C} \to \mathbb{C}$ meromorphic, and $\omega_{0,2} \in H^0(K_{\tilde{C}}^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2}$

assuming $y - y(\alpha) \sim (x - x(\alpha))^{s_\alpha / r_\alpha - 1}$

with $r_\alpha = \pm 1 \bmod s_\alpha$ and $s_\alpha \in \{1, \ldots, r_\alpha + 1\}$

Eynard-Orantin TR

by computations of periods on $\tilde{C}$

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n} F_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{a=1}^{n} d\xi_{i_a}(z_i)$$

generating series (meromorphic differential on $\tilde{C}^n$)

Kontsevich Soibelman 17

B et al. 17, 18, 20

quantum Airy structure on $\mathcal{V} = \bigoplus_{\alpha \in \mathfrak{a}} \bigoplus_{k \geq 0} \mathbb{C}.e_{\alpha,k}$

based on $\oplus_\alpha W(\mathfrak{gl}_{r_\alpha})$

3. Spectral curve description

- The theory of Airy structures proves it is well-defined, e.g. $F_{g,n}$ symmetric (previously only known for $r_\alpha = 2$)

⟶ Definition of B-model for 1d Landau-Ginzburg model

- … But also gives obstructions on local behavior (i.e. (s_α, r_α)) for well-definition/symmetry

- Extended to a description of admissible singular curves B Kramer Schüler 20

?? For each admissible curve: describe Frobenius structure on the deformation space

?? What is special geometrically for non-admissible curves

If there is an Frobenius structure, it cannot be defined as in Dubrovin theory for Hurwitz spaces

4. 4d N = 2 supersymmetric gauge theory

Donaldson 84

- $M_G^d = \left\{ \begin{array}{c} \text{Moduli space of anti-self-dual} \\ SU(r) \text{ instantons on } \mathbb{S}^4 \\ \text{framed at } \infty, \text{ instanton } \# = d \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{Algebraic } SL(r, \mathbb{C})\text{-bundles on } \mathbb{P}^2 \\ \text{with } c_2 = d \ + \text{ trivialisation on } l_\infty \end{array} \right\}$

- In N = 2 supersymmetric pure gauge theory with equivariant parmeters $\epsilon_1, \epsilon_2, (Q_i)_{i=1}^{r-1}$ for the action of $\mathcal{G} = (\mathbb{C}^*)^2 \times$ Cartan torus, the partition function reduces to "integrals" over M_G^d

$$Z_{\mathrm{Nek}}(\Lambda) = \exp\left(\sum_{g \geqslant 0} \hbar^{g-1} \mathfrak{F}_g(\Lambda, \alpha) \right)$$

$\hbar = -\epsilon_1 \epsilon_2 \to 0$

$\alpha = \epsilon_1 + \epsilon_2 \in \mathbb{C}$

Λ coupling / energy scale

- Mathematically:

$$|1^d\rangle \in IH^*_{\mathcal{G}}(\widetilde{M}_G^d) \quad \rightsquigarrow \quad |\mathbf{1}\rangle = \sum_{d \geqslant 0} \Lambda^{dr} |1^d\rangle \quad \rightsquigarrow \quad Z_{\mathrm{Nek}}(\Lambda) = \langle \mathbf{1} | \mathbf{1} \rangle$$

Fundamental class of a (partial) compactif. — Gaiotto vector — Nekrasov partition function

Uhlenbeck, Donaldson, …

4. 4d N = 2 supersymmetric gauge theory

Alday-Gaiotto-Tachikawa conjectures : relations to $W(\mathfrak{sl}_r)_{\mathfrak{c}}$- conformal blocks

$\mathfrak{c} = r - 1 - r(r^2 - 1)\hbar^{-1}\alpha^2$

The mathematical theorem incarnating this is
(here for $\mathfrak{gl}_r$)

Okounkov, Maulik 12
Schiffmann, Vasserot 13
Braverman, Finkelberg, Nakajima 14

- $\mathcal{H} = \bigoplus_{d \geqslant 0} \mathrm{IH}^*_{\mathcal{G}}(\widetilde{\mathrm{M}}^d_{\mathrm{G}})$ is a Verma module for $W(\mathfrak{gl}_r)_{\mathfrak{c}}$ (highest weight vector $|0\rangle$)
- $W^i_k|\mathbf{1}\rangle = \delta_{i,r}\delta_{k,1}\Lambda^{rd}\,|\mathbf{1}\rangle$ for all $i \in \{1,\ldots,r\}$ and $k > 0$ (Whittaker vector)
- An explicit description of the intersection pairing in $\mathcal{H}$

→ W^i_k can be represented as differential operators $\in \mathcal{D}^{\hbar}_{\mathcal{V}}$
with $\mathcal{V} = \mathbb{C}^r[[z]]$

4. 4d N = 2 supersymmetric gauge theory

Theorem 6 B Bouchard Chidambaram Creutzig 21

$$\boxed{\Lambda = \hbar^{\frac{r}{2}}\widehat{\Lambda}}$$

$(W^i_k - \hbar^{\frac{r}{2}}\widehat{\Lambda}\delta_{i,r}\delta_{k,1})^{k>0}_{1\leqslant i\leqslant r}$ is a quantum Airy structure

and its partition function is $|\mathbf{1}\rangle = \exp\Big(\sum_{g\in\frac{1}{2}\mathbb{N},\ n>0}\frac{\hbar^{g-1}}{n!}F^{(\widehat{\Lambda})}_{g,n}\Big) \in \mathrm{Fun}_{\hbar}(\mathcal{V})$

It is computed by TR associated to

- the (unramified) spectral curve $\prod_{a=1}^r\big(y - \frac{Q_a}{x}\big) = 0$

  if $\epsilon_1 + \epsilon_2 = 0$

  periods of algebraic functions

- the non-commutative spectral curve $\prod_{a=1}^r\big(\alpha\partial_x - \frac{Q_a}{x}\big)$

  if $\epsilon_1 + \epsilon_2 \neq 0$

  (refined)

  (regular D-module on $x \in \mathbb{P}^1$)

  periods from solutions of the D-module

  (new construction of TR)

  For $r = 2$: Eynard et al. 13-19

5. Intersection theory

Virasoro constraints for GW(pt) —- Witten's conjecture

correspond to the Airy structure (r,s) = (3,2)

TR on the spectral curve $x = y^2$ computes

$$\omega_{g,n}^{(3,2)}(z_1, \ldots, z_n) = \sum_{d_1,\ldots,d_n \geqslant 0} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \prod_{i=1}^{k} \frac{(2k_i+1)!!dz_i}{z_i^{2k_i+2}}$$

is the tip of an iceberg …

5. Intersection theory

$w_{g,n}$	*TR on spectral curve*	*Partition function*	
$[1]_{\overline{\mathcal{M}}_{g,n}}$	$x = y^2$	$F^{(3,2)}$	… Kontsevich 92
Witten r-spin	$x = y^r$	$F^{(r,r+1)}$	… Milanov 16
$p_* c_{\mathrm{top}}(-R^\bullet \pi_* \mathcal{L})$ from $\overline{\mathcal{M}}^{-1/r}_{g,n}$	$x = y^{-r}$	$F^{(r,r-1)}$	$r = 2$: Norbury 17 Bouchard, B., Chidambaram, Chiodo, Tessler, Norbury, …
$\Omega^{(r,s)}_{g,n}$ **?**	$x^{r-s} y^r = 1$	$F^{(r,s)}$	BBCCN 18 B Kramer Schüler 20
open r-spin GW(pt) Pandharipande Solomon, Tessler 15- …	$y(x - y^r) = 0$	$\widetilde{F}^{(r,s)}$	$r = 2$: Alexandrov 16 all r : conjecture BBCCN 18

Expectation To each Airy structure based on a VOA computes intersection theory of a class $\Omega_{g,n}$ on moduli of curves

5. Intersection theory

Conjectural (r,s) class

For $s \in \{1, \ldots, r-1\}$ such that $r = \pm 1 \bmod s$

there exists $w_{g,n}^{(r,s)}(\mathbf{a}) \in H^{\bullet}(\overline{\mathcal{M}}_{g,n})$ indexed by $\mathbf{a} \in \{1, \ldots, r\}^n$ such that

- $$\omega_{g,n}^{(r,s)}(z_1, \ldots, z_n) = \sum_{\substack{d_1,\ldots,d_n \geqslant 0 \\ 1 \leqslant a_1,\ldots,a_n \leqslant r}} \left(\int_{\overline{\mathcal{M}}_{g,n}} w_{g,n}^{(r,s)}(\mathbf{a}) \prod_{i=1}^{n} \psi_i^{d_i} \right) \prod_{i=1}^{n} \frac{(rd_i + a_i)!^{(r)} dz_i}{z_i^{rd_i + a_i + 1}}$$
- $w_{g,n}^{(r,s)}(\mathbf{a})$ has pure dimension $\frac{s}{r}(2g - 2 + n) - \sum_{i=1}^{n} \frac{a_i}{r}$
- It is a modified CohFT (Norbury)

  $\psi_{n+1} \pi^*(w_{g,n}^{(r,s)}(\mathbf{a})) = w_{g,n+1}(s, \mathbf{a})$ under forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$
- Initial values $w_{0,3}^{(r,s)}(\mathbf{a}) = \frac{1 \mp r}{s} \delta_{a_1 + a_2 + a_3, s}$ $w_{1,1}^{(r,s)}(a) = \frac{r^2 - 1}{s} \delta_{s,a} \psi_1$
- The partition function is a tau-function of r-KdV

5. Intersection theory

Theorem Eynard 12 | Chekhov Norbury 17 { B Kramer Schüler 20 modulo existence of $\Omega^{(r,s)}$

For any smooth admissible spectral curve can be constructed $w_{g,n} \in H^{\bullet}(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{A}^{*\otimes n}$ with $\mathcal{A} = \mathrm{span}_{\mathbb{C}}(\gamma_\alpha = \text{thimble for } x : \tilde{C} \to \mathbb{C}\}$ such that

$$\int_{z_i \in \gamma_{\alpha_i}} \omega_{g,n}(z_1,\ldots,z_n) \prod_{i=1}^{n} e^{-\mu_i x(z_i)} = \left(\prod_{i=1}^{n} C(\mu_i)\right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{w_{g,n}\left(\otimes_{i=1}^{n} e_{\alpha_i}\right)}{1 - \mu_i \psi_i}$$

- Givental group $\subseteq$ {symmetries of Airy structures}

  = {deformation of spectral curves respecting local behavior}

$\longrightarrow$ retrieves TR for semisimple CohFTs of Dunin-Barkowski, Orantin, Shadrin, Spitz 12
for smooth spectral curves with simple ramifications

Relies on Teleman's classification of semisimple CohFTs

- Many applications (e.g. ELSV-like formulas in Hurwitz theory)

6. Open problem

Establish a general picture :

(Equivariant) LG potential
with isolated critical point

Fan-Jarvis-Ruan-Witten theories
(generalisation of Witten r-spin classes)

for ADE

???

VOA + twisted representation
+ ideal of constraints

Integrable hierarchies (Drinfeld-Sokolov, …)

i.e. Enlarge our understanding of building blocks for topological field theories
—- thanks to symmetries, intersection-theoretic representations of TR correlators

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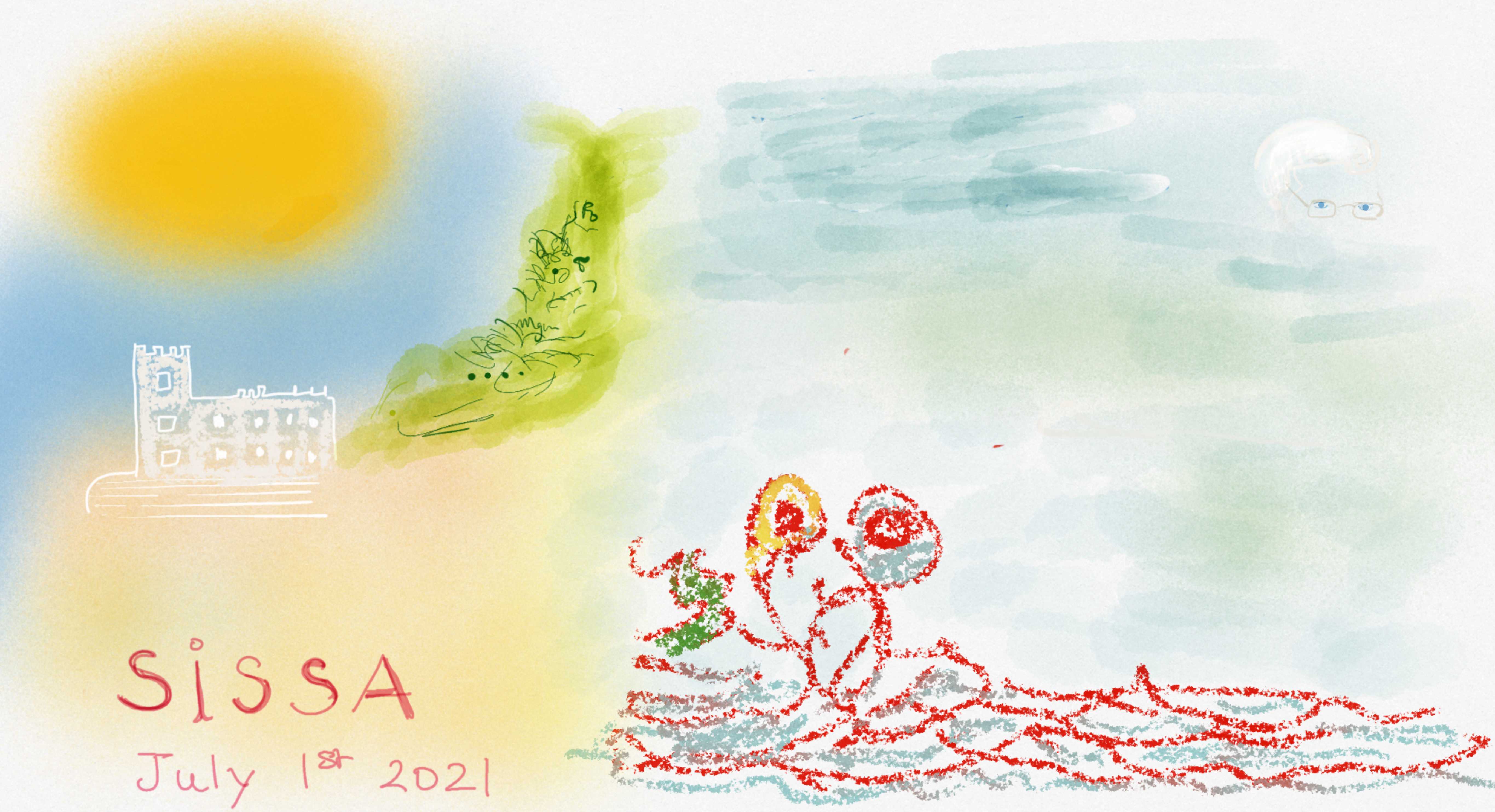

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