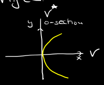
W algebras : between representation theory, 2d geometry & 5d gauge theories

I - QUANTIZATION OF LAGRANGIANS in T^*V

- V vector space $\quad (x_i)_{i\in I}$ linear coordinates.

  $T^*V = V \oplus V^*$ $\quad (x_i, y_i)_{i\in I}$ $\quad \{y_i, x_j\} = \delta_{ij} \quad \forall i,j \in I.$

  Lagrangian (submanifolds) in T^*V.
  tangent to 0 - section $\{x_i = 0 \ \forall i\}$
  at the origin



$\Lambda = \{ \lambda_i = 0 \quad \forall i \in I \}$

Lagrangian $\Longrightarrow$ $\{\lambda_i, \lambda_j\}|_\Lambda = 0$

ex $\quad \lambda_i = y_i - \sum_{a,b\in I}' \frac{1}{2} \underset{\smile}{A^i_{ab}} x_a x_b + \underset{\smile}{B^i_{ab}} x_a y_b + \frac{1}{2} \underset{\smile}{C^i_{ab}} y_a y_b.$ $\quad$ quadratic case

$$\underbrace{\{\underset{\smile}{\lambda_i}, \underset{\smile}{\lambda_j}\}}_{4-2} = \sum_{a\in I} \underset{\hookrightarrow \in \mathbb{C}}{\textcircled{f^a_{ij}}} \lambda_a$$

- quantization $\quad \mathbb{C}[T^*V] \rightsquigarrow$ $\quad$ Weyl quantization $\mathbb{C}[(x_i)_{i\in I}, (\hbar \partial_{x_i})_{i\in I}, \hbar] = \mathcal{D}^\hbar_V$

  Good quantization $\quad \{\lambda_i = 0 \ \forall i\}$ $\quad \rightsquigarrow \quad \hat{\lambda}_i \in \mathcal{D}^\hbar_V$

  $\{\lambda_i, \lambda_j\} = \sum_a \underset{\smile}{f^a_{ij}} \lambda_a$ $\quad \rightsquigarrow \quad [\hat{\lambda}_i, \hat{\lambda}_j] = \hbar \sum \underset{\underset{\mathcal{D}^\hbar_V}{\Uparrow}}{\textcircled{f^a_{ij}}} \hat{\lambda}_a$ $\quad$ ✱

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deg 2 $\quad f^a_{ij} \in \mathbb{C}$ $\qquad f^a_{ij} \in \mathbb{C}.$

$\underset{i\in I}{\text{span}}(\hat{\lambda}_i)$ is a Lie alg.

f = str constants.

$xx\partial\partial$
$\partial\partial xx \rightsquigarrow \hbar x\partial, \hbar^2$

$$\hat{\lambda}_i = \underset{\smile}{\hbar \partial_{x_i}} - \sum_{a,b\in I}' \frac{1}{2} A^i_{ab} x_a x_b + B^i_{ab} \underset{\smile}{x_a \hbar \partial_{x_b}} + \frac{1}{2} C^i_{ab} \hbar\partial_{x_a} \hbar\partial_{x_b}$$
$$- \boxed{\hbar D^i}$$
ambiguity of quantization

quadratic rel on (A, B, C, D).

Def (Kontsevich - Soibelman)
A quantum Airy structure (QAS)
is that data of $\hat{\lambda}_i \in \mathcal{D}^\hbar_V$ $\quad i \in I$ $\qquad -[x_i, \hbar\partial_{x_j}] = \delta_{ij} \hbar$

① $\hat{\lambda}_i = \hbar \partial_{x_i}$ + higher degree $\qquad$ deg $x_i = 1$
deg $\hbar = 2$

✱ ② $[\hat{\lambda}_i, \hat{\lambda}_j] \subseteq \hbar \, \mathcal{D}^\hbar_V \cdot \underset{i\in I.}{\text{span}}(\hat{\lambda}_i)$
compatibility condition.

QAS $\rightsquigarrow$ there ② is a wave function $Z((x_i)_{i\in I}, \hbar)$
$\forall i \in I \quad \hat{\lambda}_i Z = 0$ $\qquad \frac{1}{\hbar}\underset{\smile}{xxx} \Rightarrow 1$

① unique if $\quad Z = \exp\left(\sum_{g\geq 0}' \hbar^{g-1} F_g(x)\right).$

$\left[\begin{array}{l} \underset{\sim}{F_0} \text{ has no linear, quadratic terms} \\ F_g \text{ has no constant.} \end{array}\right.$
(only terms on positive degrees in $\ln Z$)

get
$$\partial_{x_i} F_g = \sum_{a,b} \frac{1}{2} A^i_{ab} x_a x_b + \delta_{g,1} \overset{\circ}{D^i}$$
$$+ \sum_{a,b} B^i_{ab} x_a \partial_{x_b} F_g + \frac{1}{2} C^i_{ab} \left(\partial_{x_a} \partial_{x_b} F_{g-1} + \sum_{h+h'=g}' \partial_{x_a} F_h \cdot \partial_{x_b} F_{h'} \right)$$

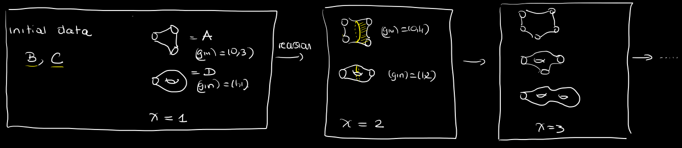
$$\textcircled{\hbar^{g-1} F_g} = \sum_{k_1,\dots,k_n} \hbar^{g-1} F_{g,n}[k_1 \dots k_n] \frac{x_{k_1} \cdots x_{k_n}}{n!}$$ deg $2g-2+n$

$$\rightsquigarrow \underline{F_{g,n}[i, \underline{k_2 \dots k_n}]} = \left| \begin{array}{ll} \textcircled{A^i_{k_2,k_3}} & (g,n) = (0,3) \\ \textcircled{D^i} & (g,n) = (1,1) \\ B \cdot F_{g,n-1} + \frac{1}{2} C \Big(F_{g-1,n+1} & \\ \qquad + \sum_{\substack{J\cup J' = \{k_2 \dots k_n\} \\ h+h'=g}} F_{h,1+|J|} \cdot F_{h',1+|J'|} & \text{otherwise.} \end{array} \right.$$

$\textcircled{\text{topological}}$ recursion on $\chi = 2g-2+n > 0$: unique solution

g genus
n # boundaries
of a surface ? $\qquad \chi = -$Euler caracteristic.

$F_{g,n}$ =



Slogan many pb maths counting surfaces phys. (topological) string theory + gauge theories.

where the wave function of a suitable QAS solves your problem $F_{g,n}$ g = genus n = # boundaries.

II W-ALGEBRAS ARE SOURCES OF QAS

QFT space of states } representation.
symmetries algebraic structure

QAS give algebra of symmetries for certain QFTs.

- Free boson on circle (Heisenberg algebra) $(J_n)_{n\in\mathbb{Z}}$ $[J_n, J_m] = n\delta_{n+m,0}$

  $J_{n>0}$: annihilation
  $J_{n<0}$: create

- conformal algebra (VOA) at least Vir_c ⟶ $(L_n)_{n\in\mathbb{Z}}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}\, n(n^2-1)$$

- W algebras (for any simple Lie algebra $\mathfrak{g}$). examples of VOAs.

$\mathrm{Vir}_c = \mathcal{W}_c(\mathfrak{sl}_2)$ $L(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$

$W(\mathfrak{sl}_3)$: Zamolodchikov 83 $W^2(z)$, $W^3(z)$, ⋮

$W(\mathfrak{sl}_r), W(\mathfrak{gl}_r)$ $W^r(z)$.

- Free field representations $M = \mathbb{C}[[x_1, x_2, \dots, \hbar]]$

$$J_n = \begin{cases} \underline{\hbar\partial_{x_n}} & n > 0 \\ 0 & n = 0 \\ \underline{-nx_{-n}} & n < 0. \end{cases}$$

$$L(z) = \boxed{:}\, \underset{\uparrow}{J^1(z)}\, \underset{\uparrow}{J^2(z)}\, \boxed{:} = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$$

normal order

two ind copies of Heisenberg algebra

$:J_{-7}J_4: = J_{-7}J_4$
$:J_4J_{-7}: = J_{-7}J_4$

For $W(\mathfrak{gl}_r)$ $W^i(z) =\, :$ i-th elementary symmetric $(J^1(z) \dots J^r(z))\,:$

$i = 1, \dots, r$ ⇝ degree i diff operator.

$\boxed{c = r}$

- 1) Twist
$\sigma \in \mathfrak{S}_r$ $\qquad \sigma(J^a_n) = \exp\left(\frac{2i\pi n}{r}\right) J^{a+1}_n$
$\sigma = (1 \cdots r)$

$\underline{r=2}$

$$\begin{cases} J_+(z) = \sum_{n\in\mathbb{Z}} \frac{\boxed{J_{2n}}}{z^{n+1}} \\ J_-(z) = \sum'_{n\in\mathbb{Z}+\frac{1}{2}} \frac{J_{2n}}{z^{n+1}} \end{cases} \rightsquigarrow L(z) = \frac{1}{2} :J_+^2(z): - \frac{1}{2} :J_-^2(z): = \sum_{n\in\mathbb{Z}} \frac{\boxed{L_n}}{z^{n+2}}$$

Heisenberg

representation of $\mathrm{Vir}_{c=2}$.

- 2 Shift $\qquad X_s \longrightarrow X_s - 1$
for some $s > 0$.

- 3 Select a subset of modes. $\qquad n \geqslant -(s-2)$.

$1 + 2 + 3 \rightsquigarrow$ QAS $\qquad V = \bigoplus_{m>0} \mathbb{C}\cdot e_m \qquad (x_m)_{m>0}$ linear coord.

$\underline{r=2}$, $\sigma = (12)$
$s = 3$

$$L_{-1} = \hbar \partial_{x_1} - \frac{1}{2} x_1^2 - \sum_{m>0}{}' \hbar\, x_{2m+1} \partial_{x_{2m+3}}$$
$$L_0 = \hbar \partial_{x_3} - \sum_{m>0}{}' (2m+1)\, x_{2m+1} \hbar \partial_{x_{2m+1}} - \frac{\hbar}{8}$$
$$L_1 = \vdots$$
$$\vdots$$

Witten conjecture / Kontsevich theorem (+ Dijkgraaf, Verlinde^2)

~ 90s.

$\mathcal{M}_{g,n} = \{$ genus g Riemann surfaces with n punctures $\}$. $\rightsquigarrow \overline{\mathcal{M}}_{g,n}$
$\quad \ni (C, p_1, \dots, p_n) \qquad$ dim $= 3g-3+n$.

$$c_1(T^*_{p_i} C) = \psi_i$$

$$F_{g,n}[2k_1+1 \cdots 2k_n+1] = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \prod_{i=1}^{n} (2k_i+1)!!$$

$$Z = \exp\left(\sum_{\substack{g,n \\ k_1,\dots,k_n}} \frac{\hbar^{g-1}}{n!} F_{g,n}[2k_1+1, \dots 2k_n+1]\, x_{2k_1+1} \cdots x_{2k_n+1} \right)$$

↗ 2d quantum gravity $\qquad L_i Z = 0 \quad \forall i \geqslant -1$
↪ GW($\{pt\}$).

Gromov-Witten theory / A-topological strings.

$\boxed{(C, p_1, \dots, p_n)} \xrightarrow{\pi} X$ target Kähler manifold

$p_i \in$ Lagrangian branes.

$$GW_{g,n}(\alpha_1, \dots, \alpha_n) = \int_{[\mathcal{M}_{g,n}(X)]} \pi^*\psi_1^{k_1} \cdots \pi^*\psi_n^{k_n} \, ev_1^*\alpha_1 \cdots ev_n^*\alpha_n$$

$$\begin{tikzcd} \mathcal{M}_{g,n}(X) \arrow[d, "\pi"] \arrow[dr, "(ev_1, \dots, ev_n)"] & \\ \mathcal{M}_{g,n} & X^n \end{tikzcd}$$

$\alpha \in H^\bullet(X)$

- $X = \mathbb{P}^1$

- $X =$ toric CY3 (complex 3 manifold) $(X = \mathbb{C}^3)$.

Z_{GW} is given by the TR / QAS.

[killed by QAS $\underset{\text{basis of } H^\bullet(X)}{\bigoplus} \mathrm{Vir}_{c=1}$]

[remodeling the B-model: Bouchard Klemm, Marino, Pasquetti]

proved by Eynard Orantin 13

Liu 15-16

general expectation

$GW(X)$ $\quad H^\bullet(X) \leftrightarrow \overset{\text{VOA}}{\{\text{primary fields}\}}$.

III RELATION TO MIRROR SYMMETRY

geometric invariants (counting pb) = periods (integrals along cycles) diff forms on algebraic variety.

A- side | B -model

$GW(X)$ = periods. $\hat{X}$ mirror.

Log boundary condition (branes)

$\int_\gamma \omega$ $\quad \gamma =$ Lagrangian cycle on $\hat{X}$

$GW\left(\boxed{\text{toric CY3 folds}}\right)$

toric CY.

$\cup\!|$

$\{H(e^x, e^y) = 0\}$ degeneration of torus fibration

$\swarrow x$

$\mathbb{C}^*$ mirror curve.

"TR on the mirror curve"

- $z^{1/2}$ $\quad \boxed{x = z^2}$

way to associate to "any" branched cover of complex curves.

a QAS ($\simeq \bigoplus \mathrm{Vir}_{c=1}$)

CFT

twisted bosons

∂x_k ......

$$\sum_{g\geq 0} \hbar^{g-1} \underset{\text{worldsheet}}{F_{g,n}} [k_1, \dots, k_n] = \langle 0 | J_{k_1} \cdots J_{k_n} (\ln \hat{Z}) | 0 \rangle$$

bosonic correlation function.

$M \ni \underline{1} = |0\rangle \quad |Z\rangle = \hat{Z}|0\rangle$

string theory $|0\rangle, 1$

↙ gauge th.

② × 2.

↘ CY3

GW

$\boxed{L_i} |Z_{pt}\rangle = 0 \qquad \forall i \geq -1.$

$\hbar\partial_{x_1} + \cdots$

$T^*V = V \oplus V^* = \overset{\text{lag}}{W} \oplus W^*$

translation $\quad x_i \to x_i + \boxed{t_i}$

change polarisation $\quad L_i \to \mathcal{U} L_i \mathcal{U}^{-1}$

$|Z_{pt}\rangle \mapsto \mathcal{U}|Z_{pt}\rangle.$

wave f^n.

$$\begin{tikzcd} \hbar\partial x_i \arrow[d] & x_i \arrow[d] \\ \hbar\partial x_i & x_i + \boxed{\phi_{ab}} \hbar\partial x_b \end{tikzcd}$$

$$\boxed{\mathcal{U}} |Z_{pt}\rangle \otimes \cdots \otimes |Z_{pt}\rangle = |Z\rangle$$

- matrix models
- GW(toric CY3)
- .....

- $x = z^2$ + translation + change of polarisation

$\rightsquigarrow [P(x,y) = 0]$