Intro to Categorification, Diagrammatics, and the Hecke Category

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1/18/22

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GENERATORS + RELATIONS : we love 'em

How to define an action of a group? <u>Method 1</u>: holistically. Define $g\cdot(-)$ for all g.

Check whole mult. table!

Ex: $S_n \circlearrowright \{1,2,\dots,n\}$ or on size k subsets, function spaces, etc.

Know directly how each $g \in G$ acts

Ex: Gens + Relns

$$S_n = \Big\langle s_i = (i\ i{+}1) \ \Big| \ \overset{\text{quadratic reln}}{s_i^2 = 1}\ ,\ \overbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\ ,\ s_i s_j = s_j s_i}^{\text{braid relns}} \ \text{for } |i-j|>1 \Big\rangle$$

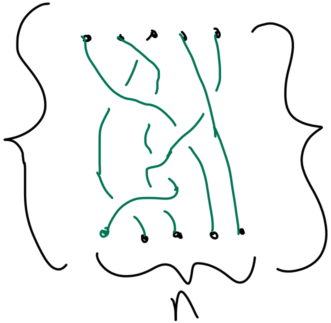
Coxeter presentation

Define action of generators + check relations

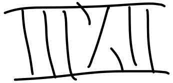
Not obvious how $g \in G$ acts... but you don't NEED to know

↳ Motivating Ex: Quantum actions of braid groups.

$Br_n =$



is generated by $f_i =$



and f_i^{-1} for $1 \le i \le n-1$

relations: braid relns (but no quadratic)

f_i acts on $V^{\otimes n}$ via particular formula, where $U_q(\mathfrak{g}) \circlearrowright V$.

Action of braid β? Trace$(\beta) \in \mathbb{Q}(q)$ is invariant of closure of braid. Useful

Gens + Relns let us define actions more easily in non-holistic circumstances, + prove isoms too.

Holistic like working w/ universal presentation, the most useless one.

$$G = \langle g \in G \mid g \cdot h = (gh) \rangle$$

How to define an action of a group? ... on a category?

$g \mapsto F_g$ a functor $\mathcal{C} \to \mathcal{C}$

want $F_g \circ F_h \cong F_{gh}$ ... canonically!

Ex: Br_n often acts on categories in geometry via spherical twists / wall crossing / etc.

- Instead of knot invariants (valued in $\mathbb{Q}(q)$) get knot homologies (valued in qVect)

  More powerful invariant, just like $H^*(X)$ vs $\chi(X)$. A complex of v.s has more info than a number.

- For a knot cobordism, get morphism b/w vector spaces.

  ↑ requires knowing braid canonically and not just up to isomorphism

Classical approach to groups $\bar{c}$ categories: Need

- F_g for $g \in G$
- $\alpha_{g,h}: F_g \circ F_g \to F_{gh}$ for $g,h \in G$

satisfying assoc

$$\begin{tikzcd} & {} \arrow[dr] & \\ F_g \circ F_h \circ F_k \arrow[ur] \arrow[dr] & \| & F_{ghk} \\ & {} \arrow[ur] & \end{tikzcd}$$

This is like the stupid universal presentation Want: effective method!

Want: • F_g $g \in$ generating set

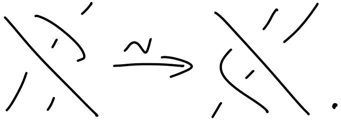
- For each relation $r: (g_1 g_2 g_3 = h_1 h_2 h_3)$ an isom

$$\alpha_r : F_{g_1} F_{g_2} F_{g_3} \xrightarrow{\sim} F_{h_1} F_{h_2} F_{h_3}$$

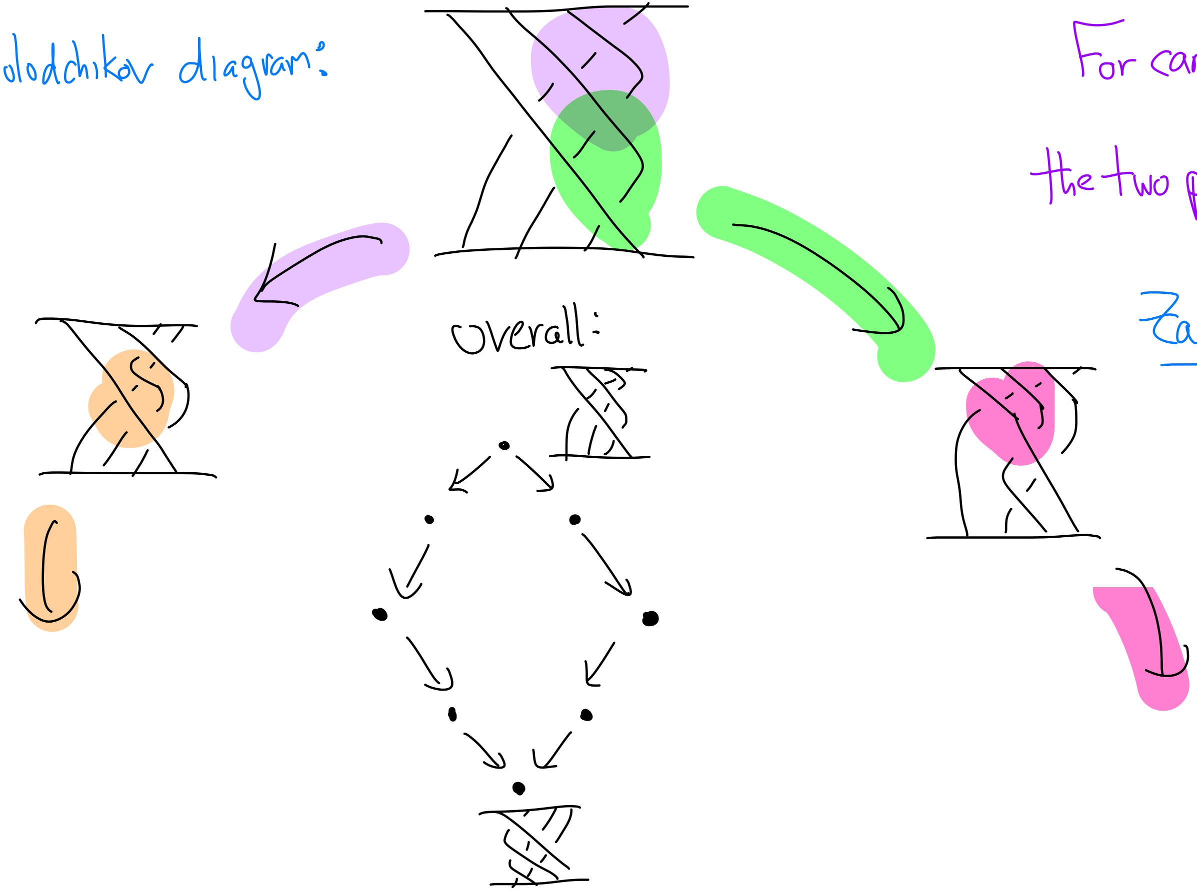
satisfying ???

what do you need to make canonical?

Called a 2-presentation of G

Ex: braid relation 

Zamolodchikov diagram:



For canonical isom
$$\begin{tikzcd}\text{top} \arrow[d, bend right] \arrow[d, bend left] \\ \text{bot}\end{tikzcd}$$

the two paths must agree

Zamolodchikov reln

Thm (E-Williamson): For an action of Br_n or S_n by functors defined via gens+relns, only need to check

- cyclicity + bridging $\longleftarrow$ Some compatibility b/w α_r and F_g^{-1}
- Zam (for $S_4 \subset S_n$) and two more ($S_3 \times S_2 < S_n$, $S_2 \times S_2 \times S_2 < S_n$)

Rephrased:

Def: For G a group, the 2-groupoid Ω_G is a monoidal category (has $\otimes$) with objects $g \in G$, satisfying $g \otimes h \simeq (gh)$, and $\mathrm{Hom}(g,h) = \begin{cases} \{id\} & \text{if } g=h \\ \emptyset & \text{else} \end{cases}$

We give a presentation of Ω_{S_n} w/ generating objects $\{s_i\}$ (generators of group)

generating morphisms $\longleftrightarrow$ relations in group

relations $\longleftrightarrow$ Zam, etc New in 2-presentation

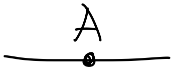
(This theorem is toy example, step en route to main topic: presentation of Hecke category)

It seems fancy but powerful tool, Diagrammatics, makes it straightforward.
Proof is elementary planar graph theory!!

Brief intro to diag. for monoidal cats) Two running examples: 1) Describing an algebra in Vect 2) ΩS_n

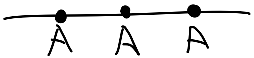
Denote an object as a labeled point on a line segment

1) $A \in \mathrm{Vect}_{\Bbbk}$



2)

$\otimes$ of objects : concat



is $A \otimes A \otimes A$

is $s \otimes t \otimes s \otimes s$

monoidal identity ⟺ no label or point needed

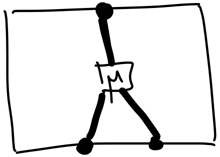
$\Bbbk$ is

b/c $k \otimes A \cong A$

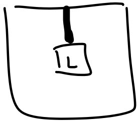
morphism encoded by rectangle with source = bottom target = top

mult

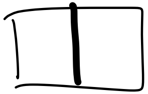
$\mu \; A\otimes A \to A$ is



unit $\Bbbk \to A$ is



$id_A : A \to A$ is

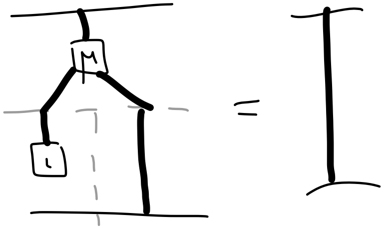


Composition: vertical stack

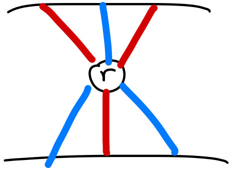
$1 \cdot a = a$

unit axiom

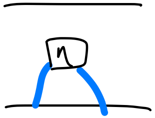
$\mu \circ (i \otimes id_A) = id_A$



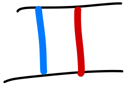
Isom $sts \to tst$ is



Isom $ss \to 1$ is

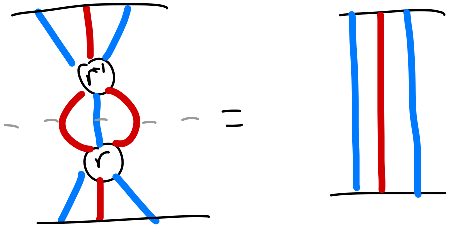


id_{st} is

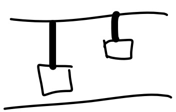
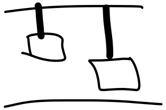


$\otimes$ horizontal concat

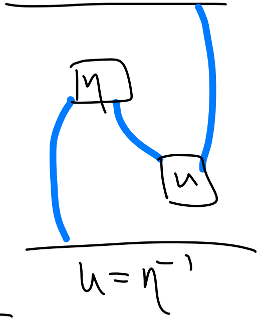
inverse Isoms



Axioms of monoidal cat: diagram/rectilinear isotopy unambiguously represents a morphism

Eg  = 

In ΩS_n have more: cyclicity



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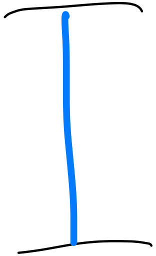

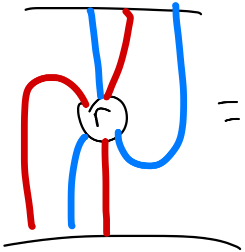

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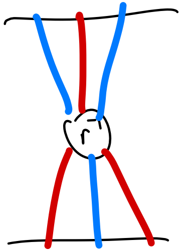
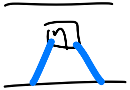
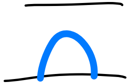
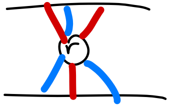
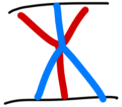


can draw 

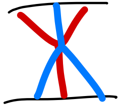
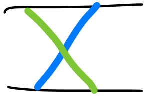
as 



as 

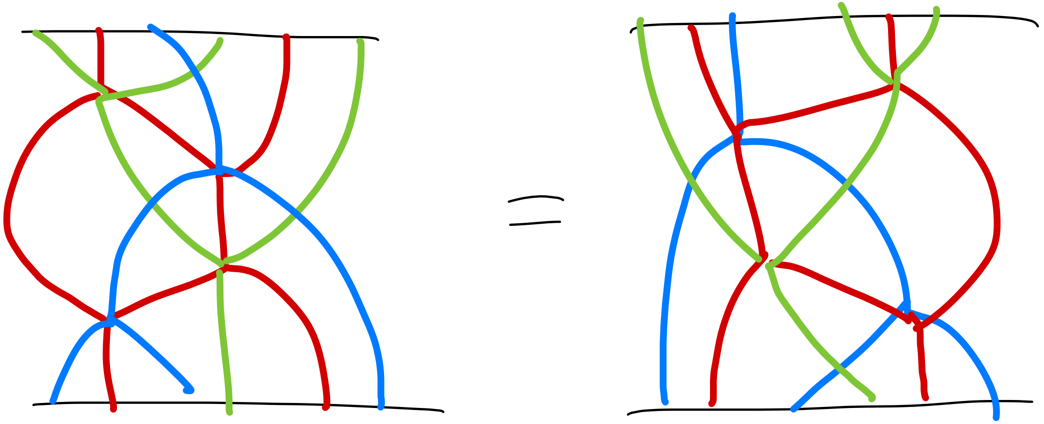
and diagram/isotopy unamb repr a morphism

Any morphism in ΩS_n represented by planar graph w/ boundary, w/ vertices

 $sts = tst$  $su = us$

Zam. in diagrams: two paths stsuts → utsutu are equal

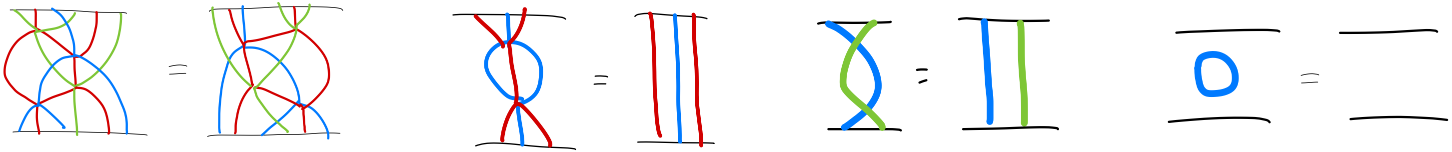
$s=s_1$ $t=s_2$ $u=s_3$



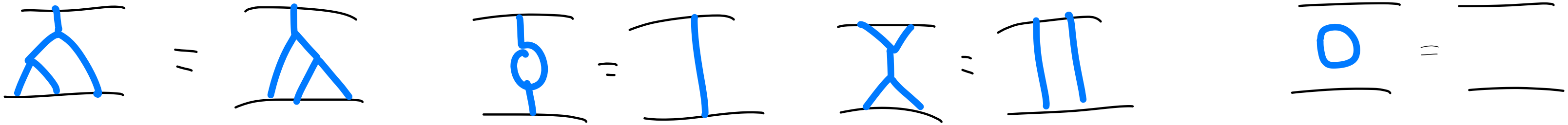
Goal for Ω_{S_n}: find enough relations so that any two graphs w/ same boundary are equal

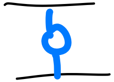
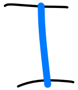
Then $\#\mathrm{Hom}(s_{i_1}s_{i_2}\ s_{i_k},\ s_{j_1}s_{j_2}\ s_{j_\ell}) = \begin{cases} 0 & \text{if elts of } S_n \text{ aren't equal (no graphs possible)} \\ 1 & \text{else, since all graphs equal} \end{cases}$

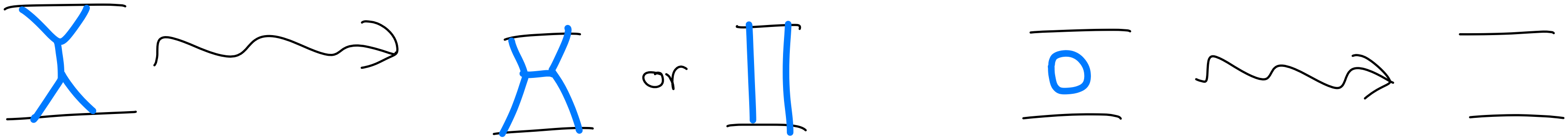
Key step in a proof (not in literature, c.f E-Khovanov) : Have relns like



Let $S = s_1$. Underlying blue graph is trivalent + relns look like this



Conversely, whenever you find  in underlying blue graph, can replace using relns with 



Easy trivalent graph theory: these operations take any trivalent graph w/ empty boundary ⇝ empty graph

SO ELIMINATE BLUE, repeat   Any graph w/ empty bdy ⇝ empty graph. $\# \mathrm{End}(1) = 1$

A general proof using topology follows ideas of Fenn + Igusa, see beautiful AlgTop textbook by Fenn.

Presentation of G ⇝ CW complex X w/ one 0-cell

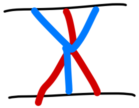
1-cells ⟷ gens

2-cells ⟷ relns

$\pi_1(X) = G$

WHAT IS $\pi_2(X)$?

word in gens ⟷ $D^1 \to X^1$



⟷ $D^2 \to X^2$

graphs w/ empty bdy ⟷ $\pi_2(X^2)$
modulo general relations

New relns like Zam ⟷ 3-cells glued in to kill $\pi_2(X^2)$

2-presentation ⟷ 3-skeleton of BG.

Our theorem in a nutshell: the Coxeter complex can be modified to provide a nice CW complex for BG

Many categorical actions of braid groups. Wouldn't it be great if they were all

SECRETLY ISOMORPHIC?

How do you show two repns are isomorphic? Holistically: define bijection. Hard when repns have different sources

Or, with structural theorems. E.g. f.d irred repns of $\mathfrak{sl}_2$ classified by highest weight.
(and isotypic repns classified by h.w. vector space)
If you match the top weight spaces, you get the rest for free

Key idea of Chuang-Rouquier (in context of $\mathfrak{sl}_2$): rigidity of categorical lifts. If you have functors lifting action of $\mathfrak{sl}_2$ (ie a weak categorification of an $\mathfrak{sl}_2$ repn), could have two non-isomorphic lifts of the same repn. But if you pin down a particular algebra of nat'l trans. then the cat'fn is unique. (Generalizations proven by Mazorchuk-Miemietz)

Many categorical actions of braid groups Wouldn't it be great if they were all

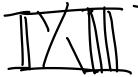
SECRETLY ISOMORPHIC ?

Reps of Br_n often factor thru $\mathbb{H}(S_n)$ Iwahori-Hecke algebra

A $\mathbb{Z}[v,v^{-1}]$-alg deforming $\mathbb{Z}[S_n]$, i.e. $\mathbb{H}(S_n)/_{(v-1)} \cong \mathbb{Z}[S_n]$.

$$\begin{tikzcd} f_i \arrow[d, mapsto] \\ h_i \end{tikzcd} \qquad \begin{tikzcd} f_i \arrow[d, mapsto] \\ s_i \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Br}_n = \langle f_i = \quad \rangle / \text{braid} \arrow[dl] \arrow[dr] & \\ \mathbb{H} = \mathbb{Z}[v,v^{-1}]\langle h_i \rangle / \text{braid} \arrow[rr] & & \mathbb{Z}[S_n] = \langle s_i \rangle / \text{braid} \end{tikzcd}$$



quadratic $(h_i+v)(h_i-v^{-1})=0$ $\qquad$ $(s_i+1)(s_i-1)=0$

Rep $\mathbb{H}$ is very similar to Rep S_n.

Given $\mathcal{C}$ a graded additive/triangulated category have Grothendieck group $[\mathcal{C}]$ the "numerical shadow", records the "size" of objects but forgets the flesh (morphisms)

$[\mathcal{C}]$ is a $\mathbb{Z}[v,v^{-1}]$-mod where $\oplus \rightsquigarrow +$ and $(1) \rightsquigarrow v$

If $Br_n \circlearrowright \mathcal{C}_1, \mathcal{C}_2$ induces $\mathbb{Z}[v,v^{-1}][Br_n] \circlearrowright [\mathcal{C}_1],[\mathcal{C}_2]$ giving same rep'n of $\mathbb{H}$, are $\mathcal{C}_1, \mathcal{C}_2$ equiv categories?? No, need more structure. Need to know about nat'l transformations b/w functors.

The Hecke Category $\mathcal{H}$ is a deformation of ΩS_n. If $\mathcal{H} \circlearrowright \mathcal{C}_1, \mathcal{C}_2$ induces $\mathbb{H} \circlearrowright [\mathcal{C}_1] \cong [\mathcal{C}_2]$ then there are criteria for deducing $\mathcal{C}_1 \cong \mathcal{C}_2$. This relates categories from:

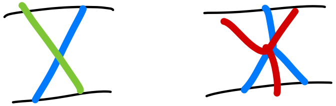
geometry of flag varieties $\longleftrightarrow$ ∞-dim rep'n theory (category $\mathcal{O}$) $\longleftrightarrow$ f.d modular rep'n theory $\longleftrightarrow$ ...

E-Khovanov-Williamson gave gens + relns for $\mathcal{H}$, building on work of Libedinsky.

Similar to diagrammatics for ΩS_n:

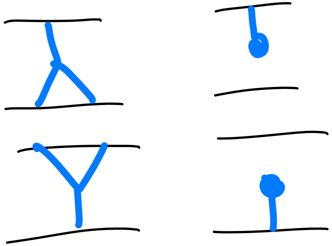
objects are sequences of simple reflections

morphisms are certain planar graphs w/ vertices



↳ but also  is an algebra object so have

actually a Frobenius algebra object:



relns: lots. Nastiest ones - Zams - are just like in ΩS_n

2^{nd} half of talk: more in depth motivation on why it is the way it is + how to prove it!

Major application: Computing multiplicities dimensions in Modular RT, previously inaccessible.

Before continuing - while I could continue purely w/ diagrammatics for many it helps to have an algebraic model to keep grounded

Model generalizes to any Coxeter group, including affine Weyl gps (the ones w/ applications!)

BUT for affine Weyl model is wrong in finite char, need diagrammatics!!

$W = S_n \circlearrowright \mathfrak{h} = \mathrm{Span}_{\Bbbk}\{x_1, \ , x_n\}$ $\quad s_i = (i \ \ i+1)$ is reflection $\perp$ to $x_i - x_{i+1} = \alpha_i$

$W \circlearrowright R = \Bbbk[x_1, -, x_n]$, $\deg x_i = 2$. Let $R^i := R^{s_i} = \{f \in R \mid s_i f = f\}$

Thm: $R^i \subset R$ is a Frobenius extension, i.e. 1) R free / R^i finite rank (rank = 2)

2) $\partial_i : R \to R^i$, R^i-linear

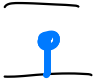
$\partial_i(f) = \frac{f - s_i f}{\alpha_i}$. "Take $\alpha_i^\vee$ $\mathfrak{h} \to \Bbbk$ and extend to R via Leibniz rule"

3) $(f, g) \mapsto \partial_i(fg)$ has dual bases

Cor. The R-bimodule $B_i := R \otimes_{R^{s_i}} R(1)$ is a Frob alg object in R-bimodules.

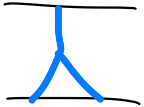
$\Longleftrightarrow$ 4 structure maps

$$\begin{tikzcd} R & R \\ R \otimes_{R^s} R(1) \arrow[u, "\text{mult}"] & B_s \arrow[u] \end{tikzcd}$$



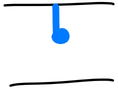
deg $= +1$

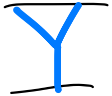
$$\begin{tikzcd} B_s \arrow[r, phantom, "="] & R \otimes_{R^s} R(1) \\ B_s \otimes B_s \arrow[u] \arrow[r, phantom, "="] & R \otimes_{R^s} R \otimes_{R^s} R(2) \arrow[u, "\text{apply } \partial_i \text{ to middle}"'] \end{tikzcd}$$



deg $= -1$

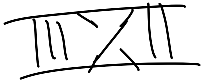
etcetera

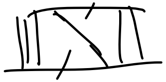




deg: $+1$ $\quad -1$

Aside: Br_n action comes from $\otimes$ w/ complexes $F_i = \left(B_i \xrightarrow{} R(1)\right)$ $\quad F_i^{-1} = \left(R(-1) \xrightarrow{} B_i\right)$





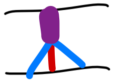
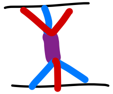
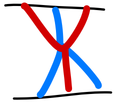
algebraic Hecke category

$\shortparallel$

Def Soergel bimodules are summands of (sums of shifts of) $\otimes$ of various B_i.

Ex: $B_{i+1} \otimes B_i \otimes B_{i+1} \overset{\oplus}{\supset} \underset{\overset{\shortparallel}{B_{s_i s_{i+1} s_i}}}{R \otimes_{R^{i,i+1}} R(3)} \overset{\oplus}{\subset} B_i \otimes B_{i+1} \otimes B_i$

In general they're mysterious! Even the map $B_sB_tB_s \longrightarrow B_{sts}$ is ugly when written in formulas for polynomials! The worst exercise...

This ugly map is abstracted into a lovely diagram  where  $=:$ 
and the relations it satisfies are pretty

MUCH EASIER TO COMPUTE w/ DIAGRAMS.

Important theme #1: When doing gens+relns, only try to use easy + combinatorial objects.

$B_sB_tB_sB_uB_vB_tB_s = BS(stsuvts)$ is easy $\rightsquigarrow$ Homs between these are comprehensible

$\cup\oplus$

P $B_{stsuvts}$ some mysterious summand, ! summand not in shorter expressions. Depends on characteristic! $\rightsquigarrow$ Homs b/w these are unknown +not flat over $\mathbb{Z}$!

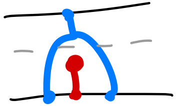
same for any reduced expression

$\mathcal{H}$ is diag description of Homs b/w $BS(s_1 s_2 \; s_d)$ for <u>all</u> sequences   BSBim ⊂ R-bim

<u>Idea 2</u>: Non-reduced sequences have no new summands <u>Redundant</u>!

Should we ignore them? DEFINITELY NOT.

1) Monoidal structure enables diagrammatics. Can't have  without 

2) Enables better factoring of morphisms. $B_sB_tB_s \cong B_{sts} \oplus B_s$ What is projection map $B_sB_tB_s \to B_s$?  factors thru B_sB_s.

REDUNDANCY IS USEFUL. (People are too obsessed with <u>minimal</u> projective generators.)

It seems silly, but Ideas 1+2 (Soergel+Kuperberg '90s, Khovanov, Lauda, Rouquier, Webster '00s) led to a major revolution

Relationship to Ω_{S_n}: Let R_w be the R-bimodule, $\cong R$ as left module, right action is

$$m \cdot r = w(r) m \qquad \text{Then } R_w \otimes R_x \cong R_{wx} \qquad \text{and} \qquad \mathrm{Hom}(R_w, R_x) = \begin{cases} R & w = x \\ 0 & \text{else} \end{cases}$$

$$1 \otimes 1 \mapsto 1$$

(if $W \curvearrowright \mathfrak{h}$ faithful)

These standard bimodules form a full subcat $\mathrm{StdBim} \subset R\text{-bim}$, isom to the R-linear version of the 2-groupoid Ω_{S_n}. They're NOT Soergel bimodules though

$$0 \to R_s(-1) \to B_s \xrightarrow{\text{mult}} R(1) \to 0$$

sub but not summand

$$0 \to R(-1) \to B_s \to R_s(1) \to 0$$

$$f \otimes g \mapsto f s(g)$$

Soergel bimodules have *Standard filtrations*

Let $Q = ff(R)$. Then over Q, $B_s \otimes_R Q \cong Q \oplus Q_s$

Std filt splits after localizing

Idea: BSBim is a generically semisimple monoidal cat.

$$\mathrm{BSBim} \hookrightarrow \mathrm{BSBim} \otimes_R Q \cong \Omega_{S_n}(Q)$$

General tricks for presenting these!

$[\mathcal{H}] \cong \mathbb{H}$

$[B_s] \mapsto b_s = h_s + v \xmapsto{v=1} 1+s \in \mathbb{Z}[S_n]$

$(1+s)^2 = 2(1+s)$

$b_s^2 = (v+v^{-1})\, b_s$

$B_s \otimes B_s \cong B_s(1) \oplus B_s(-1)$

<u>Idea</u>: Categorification equips groth gp with a <u>Hom Form</u>

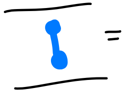
$([M],[N]) = \operatorname{gdim} \operatorname{Hom}(M,N)$ or alternatively $\operatorname{grank}_R \operatorname{Hom}(M,N)$

Hom spaces free as R-mod

Pairing on $\mathbb{H} \overset{v=1}{\rightsquigarrow}$ orthonormal product on $\mathbb{Z}[S_n]$ $=$ Hom form for Ω_{S_n}

$(w,x) = \delta_{wx}$

Action comes from

 = mult by α_s

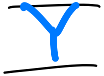
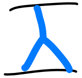
Two key Uses: 1) Inventing Rels

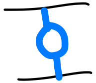
2) Finding a basis

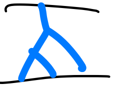
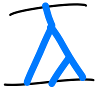
Inventing Gens + Relns from Hom form:

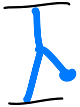
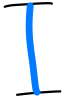
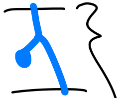
<u>Ex</u>: $(b_s, b_s) = 1 + v^2$ so $\mathrm{End}(B_s) \cong R \oplus R(-2)$ as R-mod

In deg 0 have identity. In deg −2 have nothing!

$(b_s, b_sb_s) = v^{-1} + \dots$ so ∃ map of degree −1, whence . Similarly ∃ 

then  $= 0$ b/c deg $= -2$

$\Big\{$  ,  $\Big\}$ both deg −2 in $\mathrm{Hom}(B_sB_sB_s, B_s)$ w/ gdim $= v^{-2} + \dots$ So must be lin dep!

Similarly deduce $\Big\{$  ,  ,  $\Big\}$ all colinear.

The <u>form</u> of all relations is determined this way... but not the coeffs. Can test coeffs by evaluating on bimodules

Constructing a basis

A basis for Homs in $\mathcal{S}_{S_n}$ is easy. Artin-Wedderburn...

(this basis won't lift over R b/c not semisimple there!)

Ex: Over $\mathbb{Q}$, $B_s \cong \mathbb{Q} \oplus \mathbb{Q}_s$ (ignore grading shifts)

$\mathbb{Q}_w \otimes B_s \cong \mathbb{Q}_w \oplus \mathbb{Q}_{ws}$ so need projection maps $\mathbb{Q}_w B_s \to \mathbb{Q}_w$ and $\mathbb{Q}_w B_s \to \mathbb{Q}_{ws}$

(+ inclusion maps)

If you have these for all w, s then can iteratively construct a basis for all homs over $\mathbb{Q}$

Ex $\operatorname{Hom}(B_sB_tB_s, B_s) = ?$

$$B_sB_tB_s = (\mathbb{Q}\oplus\mathbb{Q}_s)(\mathbb{Q}\oplus\mathbb{Q}_t)(\mathbb{Q}\oplus\mathbb{Q}_s) = \mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}_s\oplus\mathbb{Q}_s\oplus\mathbb{Q}_t\oplus\mathbb{Q}_{st}\oplus\mathbb{Q}_{ts}\oplus\mathbb{Q}_{sts}$$

$B_s = \mathbb{Q}\oplus\mathbb{Q}_s$ $\quad$ dim Hom = 4

have proj maps
incl

$$\begin{tikzcd} B_s = \mathbb{Q}_1 B_s \arrow[r, two heads, shift right] \arrow[dr, two heads, shift right] & \mathbb{Q} \arrow[l, shift right] \\ & \mathbb{Q}_s \arrow[ul, shift right] \end{tikzcd}$$

To get proj from $B_sB_tB_s$ to $(\mathbb{Q}_s)(\mathbb{Q})(\mathbb{Q}_s) = \mathbb{Q}$, do

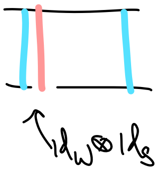
$$\mathbb{Q}_1 B_s B_t B_s \twoheadrightarrow \mathbb{Q}_s B_t B_s \twoheadrightarrow \mathbb{Q}_s B_s \twoheadrightarrow \mathbb{Q}$$

(hooray for $\otimes$)

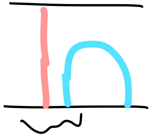
Now compose $B_sB_tB_s \Rightarrow Q \hookrightarrow B_s$ to get a matrix entry in A-W basis.

Note: We can construct all $Q_wB_s \to Q_w$ or $_{ws}$ in ΩS_n easily. Let's do $\Rightarrow Q_{ws}$ via $\otimes Q_s$ $(\subset B_s)$

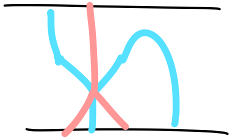
if $ws > w$



if $ws < w$



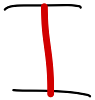
w has red exp ending in s

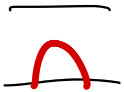


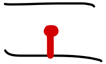
← may need to apply braid rel'n isoms to reach desired red exp first.

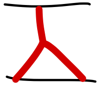
Problem: A-W basis doesn't lift over R

Solution: Modify in upper triangular fashion until you find a basis which does

Now in $\mathcal{H}$:  $\overset{\otimes_R Q}{\rightsquigarrow}$ $\begin{array}{c|cc} & Q & Q_s \\ \hline Q & 1 & 0 \\ Q_s & 0 & 1 \end{array}$ agrees w/ $id_{Q_s} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ modulo l.t. in p.o. on matrix entries

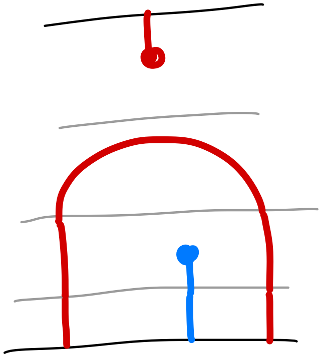
 $\rightsquigarrow$ $\begin{array}{c|cccc} & QQ & QQ_s & Q_sQ & Q_sQ_s \\ \hline Q & \alpha_s^{-1} & 0 & 0 & \alpha_s^{-1} \end{array}$ agrees w/ $(0\ 0\ 0\ 1)$ up to unit

 $\begin{array}{c|cc} & Q & Q_s \\ \hline Q & 1 & 0 \end{array}$

 $\begin{array}{c|c} & Q_sQ \\ \hline Q_s & \alpha_s^{-1} \end{array}$

FOUR ELEMENTARY PROJECTIONS

So $B_sB_tB_s \to Q \to B_s$ lifts to



which agrees w/ matrix entry met

Result A combinatorial algorithm for morphisms in $\mathcal{H}$ which become a basis of Ω_{S_n} after $(-)\otimes_R Q$

Double Leaves Basis

(Libedinsky for bimodules)

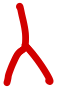
Ex basis for $\mathrm{Hom}(B_sB_s, B_s)$

4 dim 4 entries in matrix are zero
by orthogonality of Q, Q_s

$$\begin{array}{c|cccc} & QQ & QQ_s & Q_sQ & Q_sQ_s \\ \hline Q & \alpha_s & 0 & 0 & 0 \\ Q_s & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c|cccc} & QQ & QQ_s & Q_sQ & Q_sQ_s \\ \hline Q & 1 & 0 & 0 & \alpha_s^{-1} \\ Q_s & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c|cccc} & QQ & QQ_s & Q_sQ & Q_sQ_s \\ \hline Q & 1 & 0 & 0 & 0 \\ Q_s & 0 & 1 & 0 & 0 \end{array}$$



$$\begin{array}{c|cccc} & QQ & QQ_s & Q_sQ & Q_sQ_s \\ \hline Q & \alpha_s^{-1} & 0 & 0 & \alpha_s^{-1} \\ Q_s & 0 & \alpha_s^{-1} & \alpha_s^{-1} & 0 \end{array}$$

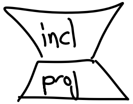
More on localization techniques:

E-W '20

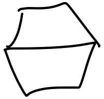
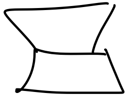
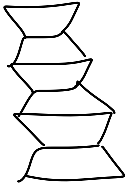
Above argument shows double leaves are linearly indep because they go to a basis

Do they span? How do you simplify ANY crazy diagram into these?

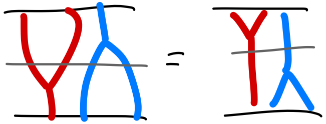
Long story short: Another general argument Essentially rewriting theory

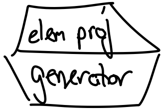
Double leaves always look like this 

← narrowest in middle

If you can take  and rewrite as  then inductively you can handle  anything

Monoidal structure makes it easier b/c so many things commute!



Reduce to finitely many cases: 

Work in progress: Manual/Manifesto so others can USE rather than recreate these nasty proofs

THANKS