Bilinear discretization of quadratic vector fields: integrability and geometry

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HU Seminar “Algebra, Geometry and Physics”
Berlin, 27.04.2021

Discretization in
Geometry and Dynamics
SFB Transregio 109

Part 1. Generalities

The problem of integrable discretization. Hamiltonian approach (Birkhäuser, 2003)

Consider a completely integrable flow

$$\dot{x} = f(x) = \{H, x\} \qquad (1)$$

with a Hamilton function H on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot,\cdot\}$. Thus, flow (1) possesses sufficiently many functionally independent integrals $I_k(x)$ in involution.

The *problem of integrable discretization*: find a family of diffeomorphisms $\mathcal{P} \to \mathcal{P}$,

$$\widetilde{x} = \Phi(x; \epsilon), \qquad (2)$$

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:

1. The maps (2) *approximate* the flow (1):

$$\Phi(x;\epsilon) = x + \epsilon f(x) + O(\epsilon^2).$$

2. The maps (2) are *Poisson* w. r. t. the bracket $\{\cdot,\cdot\}$ or some its deformation $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
3. The maps (2) are *integrable*, i.e. possess the necessary number of independent integrals in involution, $I_k(x;\epsilon) = I_k(x) + O(\epsilon)$.

While integrable lattice systems (like Toda or Volterra lattices) can be discretized in a systematic way (based, e.g., on the r-matrix structure), there is no systematic way to obtain *decent* integrable discretizations for integrable systems of classical mechanics.

Missing in the book: Hirota-Kimura discretizations

- R.Hirota, K.Kimura. *Discretization of the Euler top.* J. Phys. Soc. Japan **69** (2000) 627–630,
- K.Kimura, R.Hirota. *Discretization of the Lagrange top.* J. Phys. Soc. Japan **69** (2000) 3193–3199.

Reasons for this omission: discretization of the Euler top seemed to be an isolated curiosity; discretization of the Lagrange top seemed to be completely incomprehensible, if not even wrong.

Renewed interest stimulated by a talk by T. Ratiu at the Oberwolfach Workshop "Geometric Integration", March 2006, who claimed that HK-type discretizations for the Clebsch system and for the Kovalevsky top are also integrable.

Hirota-Kimura or Kahan?

- W. Kahan. *Unconventional numerical methods for trajectory calculations* (Unpublished lecture notes, 1993).

$$\dot{x} = Q(x) + Bx + c \quad \rightsquigarrow \quad (\widetilde{x} - x)/\epsilon = Q(x, \widetilde{x}) + B(x + \widetilde{x})/2 + c,$$

where $B \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$, each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *quadratic* form, and $Q(x, \widetilde{x}) = (Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x}))/2$ is the corresponding symmetric *bilinear* function. Thus,

$$\dot{x}_k \rightsquigarrow (\widetilde{x}_k - x_k)/\epsilon, \quad x_k^2 \rightsquigarrow x_k \widetilde{x}_k, \quad x_j x_k \rightsquigarrow (x_j \widetilde{x}_k + \widetilde{x}_j x_k)/2.$$

Linear w.r.t. $\widetilde{x}$, therefore defines a *rational* map $\widetilde{x} = \Phi_f(x, \epsilon)$. Obvious symmetry: $x \leftrightarrow \widetilde{x}$, $\epsilon \mapsto -\epsilon$, therefore Φ_f *reversible*:

$$\Phi_f^{-1}(x, \epsilon) = \Phi_f(x, -\epsilon).$$

In particular, Φ_f is *birational*, and $\deg \Phi_f = \deg \Phi_f^{-1} = n$.

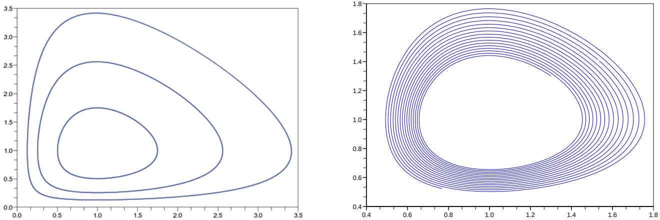
Illustration: Lotka-Volterra system

Kahan's discretization for the Lotka-Volterra system:

$$\left\{\begin{array}{l} \dot{x} = x(1-y), \\ \dot{y} = y(x-1), \end{array}\right. \quad \rightsquigarrow \quad \left\{\begin{array}{l} \widetilde{x} - x = \epsilon(\widetilde{x} + x) - \epsilon(\widetilde{x}y + x\widetilde{y}), \\ \widetilde{y} - y = \epsilon(\widetilde{x}y + x\widetilde{y}) - \epsilon(\widetilde{y} + y). \end{array}\right.$$

Explicitly:

$$\left\{\begin{array}{l} \widetilde{x} = x\,\dfrac{(1+\epsilon)^2 - \epsilon(1+\epsilon)x - \epsilon(1-\epsilon)y}{1-\epsilon^2 - \epsilon(1-\epsilon)x + \epsilon(1+\epsilon)y}, \\ \\ \widetilde{y} = y\,\dfrac{(1-\epsilon)^2 + \epsilon(1+\epsilon)x + \epsilon(1-\epsilon)y}{1-\epsilon^2 - \epsilon(1-\epsilon)x + \epsilon(1+\epsilon)y}. \end{array}\right.$$



Left: three orbits of Kahan's discretization with $\epsilon = 0.1$, right: one orbit of the explicit Euler with $\epsilon = 0.01$.

- J.M. Sanz-Serna. *An unconventional symplectic integrator of W.Kahan.* Applied Numer. Math. 1994, **16**, 245–250.

A sort of an explanation of a non-spiralling behavior: Kahan's discretization is symplectic w.r.t. $dx \wedge dy/(xy)$.

Hirota-Kimura's discrete time Euler top

$$\left\{\begin{array}{l} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{array}\right. \quad \rightsquigarrow \quad \left\{\begin{array}{l} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2). \end{array}\right.$$

Features:

- Equations are linear w.r.t. $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^{\mathrm{T}}$:

$$A(x,\epsilon)\widetilde{x} = x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix},$$

  result in a rational map, which is *reversible* (therefore birational):

$$\widetilde{x} = \Phi(x,\epsilon) = A^{-1}(x,\epsilon)x, \quad \Phi^{-1}(x,\epsilon) = \Phi(x,-\epsilon).$$

- Explicit formulas:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases}$$

where $\Delta(x,\epsilon) = \det A(x,\epsilon)$

$$= 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3.$$

- Two independent integrals:

$$I_1(x,\epsilon) = \frac{1-\epsilon^2\alpha_2\alpha_3x_1^2}{1-\epsilon^2\alpha_3\alpha_1x_2^2}, \quad I_2(x,\epsilon) = \frac{1-\epsilon^2\alpha_3\alpha_1x_2^2}{1-\epsilon^2\alpha_1\alpha_2x_3^2}.$$

- Invariant volume form:

$$\omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)}, \quad \phi(x) = 1-\epsilon^2\alpha_i\alpha_jx_k^2$$

and bi-Hamiltonian structure found in:

- M. Petrera, Yu. S. *On the Hamiltonian structure of the Hirota-Kimura discretization of the Euler top.* Math. Nachr., 2010, **283**, 1654–1663.

Hirota-Kimura's discrete time Lagrange top

Equations of motion of the Lagrange top:

$$\begin{aligned}
\dot{m}_1 &= (\alpha-1)m_2m_3+\gamma p_2, \\
\dot{m}_2 &= (1-\alpha)m_1m_3-\gamma p_1, \\
\dot{m}_3 &= 0, \\
\dot{p}_1 &= \alpha p_2m_3-p_3m_2, \\
\dot{p}_2 &= p_3m_1-\alpha p_1m_3, \\
\dot{p}_3 &= p_1m_2-p_2m_1.
\end{aligned}$$

It is Hamiltonian w.r.t. Lie-Poisson bracket of $e(3)$, has four functionally independent integrals in involution: two Casimir functions,

$$C_1=p_1^2+p_2^2+p_3^2, \quad C_2=m_1p_1+m_2p_2+m_3p_3,$$

the Hamilton function, and the (trivial) "fourth integral",

$$H_1=\frac{1}{2}(m_1^2+m_2^2+\alpha m_3^2)+\gamma p_3, \quad H_2=m_3.$$

Discretization:

$$
\begin{aligned}
\widetilde{m}_1 - m_1 &= \epsilon(\alpha-1)(\widetilde{m}_2 m_3 + m_2\widetilde{m}_3) + \epsilon\gamma(p_2+\widetilde{p}_2),\\
\widetilde{m}_2 - m_2 &= \epsilon(1-\alpha)(\widetilde{m}_1 m_3 + m_1\widetilde{m}_3) - \epsilon\gamma(p_1+\widetilde{p}_1),\\
\widetilde{m}_3 - m_3 &= 0,\\
\widetilde{p}_1 - p_1 &= \epsilon\alpha(p_2\widetilde{m}_3 + \widetilde{p}_2 m_3) - \epsilon(p_3\widetilde{m}_2 + \widetilde{p}_3 m_2),\\
\widetilde{p}_2 - p_2 &= \epsilon(p_3\widetilde{m}_1 + \widetilde{p}_3 m_1) - \epsilon\alpha(p_1\widetilde{m}_3 + \widetilde{p}_1 m_3),\\
\widetilde{p}_3 - p_3 &= \epsilon(p_1\widetilde{m}_2 + \widetilde{p}_1 m_2 - p_2\widetilde{m}_1 - \widetilde{p}_2 m_1).
\end{aligned}
$$

As usual, get an explicit birational map $(\widetilde{m},\widetilde{p}) = \Phi(m,p,\epsilon)$.

Trivial conserved quantity $m_3 = \text{const}$. Very difficult to find any further conserved quantity!

Hirota-Kimura's method for finding integrals

Incredible claim by HK: *for any initial point, there exist* $A, B, C \in \mathbb{R}$ *such that*

$$A(m_1^2 + m_2^2) + Bp_3^2 + Cp_3 = 1$$

along the orbit $\Phi^i(p, m, \epsilon)$*,* $i \in \mathbb{Z}$*.*

How one could check this? Solve the system for the unknowns A, B, C for $i = -1, 0, 1$:

$$\begin{cases} A(\widetilde{m}_1^2 + \widetilde{m}_2^2) + B\widetilde{p}_3^2 + C\widetilde{p}_3 = 1, \\ A(m_1^2 + m_2^2) + Bp_3^2 + Cp_3 = 1, \\ A(\underset{\sim}{m}_1^2 + \underset{\sim}{m}_2^2) + B\underset{\sim}{p}_3^2 + C\underset{\sim}{p}_3 = 1 \end{cases}$$

with $(\widetilde{m}, \widetilde{p}) = \Phi(m, p, \epsilon)$ and $(\underset{\sim}{m}, \underset{\sim}{p}) = \Phi^{-1}(m, p, \epsilon)$. Then check that $A, B, C = A, B, C(m, p, \epsilon)$ are conserved quantities.

Why should this work???

Hirota-Kimura sets

Definition. *For a given bijective map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, a set of functions $\Psi = (\psi_1, \ldots, \psi_\ell)$, linearly independent over $\mathbb{R}$, is called a* **HK-set**, *if for every $x_0 \in \mathbb{R}^n$ there exists a vector $c = (c_1, \ldots, c_\ell) \neq 0$, $c = c(x_0)$, such that*

$$c_1\psi_1(\Phi^i(x_0)) + \ldots + c_\ell\psi_\ell(\Phi^i(x_0)) = 0 \quad \forall i \in \mathbb{Z}.$$

For a given $x_0 \in \mathbb{R}^n$, the set $K_\Psi(x_0)$ of all vectors $c(x_0) \in \mathbb{R}^\ell$ with this property is called the null-space of the HK-set Ψ (at the point x_0). This is clearly a vector space.

Dynamical consequence. Existence of a HK-set Ψ with $\dim K_\Psi(x_0) = d$ confines orbits of Φ to $(n-d)$-dimensional invariant sets (similarly to the presence of d integrals).

From HK-sets to integrals

Proposition. *If Ψ is a HK-set for a map Φ with a d-dimensional null space then $K_\Psi(\Phi(x_0)) = K_\Psi(x_0)$*, a $Gr(d, \ell)$-valued integral.

Its Plücker coordinates are scalar integrals.

The most useful particular case:

Corollary. *Let Ψ be a HK-set for Φ with $\dim K_\Psi(x_0) = 1$ for all $x_0 \in \mathbb{R}^n$. Let $K_\Psi(x_0) = [c_1(x_0) : \ldots : c_\ell(x_0)] \in \mathbb{RP}^{\ell-1}$. Then the functions c_j/c_k are integrals of motion for Φ.*

The number of functionally independent integrals among them varies in examples (sometimes just $= 1$ and sometimes > 1).

Hirota-Kimura sets for the discrete Lagrange top

Results by Hirota and Kimura in the Lagrange top case:

Theorem. *The three sets of functions,*

$$\begin{aligned}
\Psi_1 &= (m_1^2 + m_2^2,\, p_3^2,\, p_3,\, 1),\\
\Psi_2 &= (m_1 p_1 + m_2 p_2,\, p_3^2,\, p_3,\, 1),\\
\Psi_3 &= (p_1^2 + p_2^2,\, p_3^2,\, p_3,\, 1),
\end{aligned}$$

are HK-sets for the discrete time Lagrange top with one-dimensional null-spaces, each producing one independent integral.

It follows that any orbit lies on a two-dimensional surface in $\mathbb{R}^6$ which is intersection of three quadrics and a hyperplane $m_3 = \text{const}$.

A simple integral (unnoticed by Hirota and Kimura)

Theorem. *The functions*

$$\Gamma = (\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1,\ \widetilde{m}_2 p_2 - m_2 \widetilde{p}_2,\ \widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)$$

build a HK-set for the discrete time Lagrange top with one-dimensional null-space $K_\Gamma(x) = [1 : 1 : J]$,

$$J = \frac{(2\alpha - 1) + \epsilon^2(\alpha - 1)(m_1^2 + m_2^2) + \epsilon^2\gamma(m_1 p_1 + m_2 p_2)/m_3}{1 + \epsilon^2\alpha(1-\alpha)m_3^2 - \epsilon^2\gamma p_3}.$$

Invariant volume form (unknown to Hirota and Kimura)

Theorem. *The discrete time Lagrange top possesses an invariant volume form:*

$$\Phi^*\omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\Delta(m,p)},$$

where

$$\Delta = 1 + \epsilon^2 \Delta^{(2)} + \epsilon^4 \Delta^{(4)} + \epsilon^6 \Delta^{(6)},$$

and $\Delta^{(q)}$ are polynomials of degree q in (m,p).

Further examples of integrable HK-discretizations

Overview given in

- M. Petrera, A. Pfadler, Yu. S. *On integrability of Hirota-Kimura type discretizations.* Regular Chaotic Dyn., 2011, **16**, 245–289.

1. Reduced Nahm equations.
2. Three-wave interaction system.
3. Periodic Volterra chain of period $N = 3, 4$:

$$\dot{x}_k = x_k(x_{k+1} - x_{k-1}), \quad k \in \mathbb{Z}/N\mathbb{Z}$$

4. Dressing chain with $N = 3$:

$$\dot{x}_k + \dot{x}_{k+1} = x_{k+1}^2 - x_k^2 + \alpha_{k+1} - \alpha_k, \quad k \in \mathbb{Z}/N\mathbb{Z}, \quad N \text{ odd.}$$

5. System of two interacting Euler tops.
6. Kirchhof and Clebsch cases of rigid body in an ideal fluid.

Clebsch system

Clebsch case of the motion of a rigid body in an ideal fluid:

$$\begin{aligned}
\dot{m}_1 &= (\omega_3 - \omega_2)p_2p_3, \\
\dot{m}_2 &= (\omega_1 - \omega_3)p_3p_1, \\
\dot{m}_3 &= (\omega_2 - \omega_1)p_1p_2, \\
\dot{p}_1 &= m_3p_2 - m_2p_3, \\
\dot{p}_2 &= m_1p_3 - m_3p_1, \\
\dot{p}_3 &= m_2p_1 - m_1p_2.
\end{aligned}$$

It is Hamiltonian w.r.t. Lie-Poisson bracket of $e(3)$, has four functionally independent integrals in involution:

$$I_i = p_i^2 + \frac{m_j^2}{\omega_k - \omega_i} + \frac{m_k^2}{\omega_j - \omega_i}, \quad (i,j,k) = c.p.(1,2,3),$$

and $H_4 = m_1p_1 + m_2p_2 + m_3p_3$.

Hirota-Kimura discretization of the Clebsch system

A Hirota-Kimura (or Kahan) style discretization:

$$\begin{aligned}
\widetilde{m}_1 - m_1 &= \epsilon(\omega_3 - \omega_2)(\widetilde{p}_2 p_3 + p_2 \widetilde{p}_3), \\
\widetilde{m}_2 - m_2 &= \epsilon(\omega_1 - \omega_3)(\widetilde{p}_3 p_1 + p_3 \widetilde{p}_1), \\
\widetilde{m}_3 - m_3 &= \epsilon(\omega_2 - \omega_1)(\widetilde{p}_1 p_2 + p_1 \widetilde{p}_2), \\
\widetilde{p}_1 - p_1 &= \epsilon(\widetilde{m}_3 p_2 + m_3 \widetilde{p}_2) - \epsilon(\widetilde{m}_2 p_3 + m_2 \widetilde{p}_3), \\
\widetilde{p}_2 - p_2 &= \epsilon(\widetilde{m}_1 p_3 + m_1 \widetilde{p}_3) - \epsilon(\widetilde{m}_3 p_1 + m_3 \widetilde{p}_1), \\
\widetilde{p}_3 - p_3 &= \epsilon(\widetilde{m}_2 p_1 + m_2 \widetilde{p}_1) - \epsilon(\widetilde{m}_1 p_2 + m_1 \widetilde{p}_2).
\end{aligned}$$

A birational map of $\mathbb{R}^6$ of degree 6:

$$\begin{pmatrix} \widetilde{m} \\ \widetilde{p} \end{pmatrix} = \Phi(m, p, \epsilon) = M^{-1}(m, p, \epsilon) \begin{pmatrix} m \\ p \end{pmatrix},$$

$$M(m, p, \epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

with $\omega_{ij} = \omega_i - \omega_j$. The usual reversibility:

$$\Phi^{-1}(m, p, \epsilon) = \Phi(m, p, -\epsilon).$$

Results for the discrete Clebsch system

Based on:

- M. Petrera, A. Pfadler, Yu. S. *On integrability of Hirota-Kimura type discretizations. Experimental study of the discrete Clebsch system.* Experimental Math., 2009, **18**, 223–247.
- M. Petrera, Yu. S. *New results on integrability of the Kahan-Hirota-Kimura discretizations*. - In: Nonlinear Systems and Their Remarkable Mathematical Structures, CRC Press, 2018, 94–120.

Theorem. *a) The set of functions*

$$\Psi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1, m_2 p_2, m_3 p_3, 1)$$

is a HK-set for Φ*, with* $\dim K_\Psi(m, p) = 4$*. Thus, any orbit of* Φ *lies on an intersection of four quadrics in* $\mathbb{R}^6$*.*

b) The following four are HK-sets for Φ *with one-dimensional null-spaces:*

$$\begin{aligned} \Psi_0 &= (p_1^2, p_2^2, p_3^2, 1), \\ \Psi_1 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1), \\ \Psi_2 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2 p_2), \\ \Psi_3 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3 p_3). \end{aligned}$$

There holds: $K_\Psi = K_{\Psi_0} \oplus K_{\Psi_1} \oplus K_{\Psi_2} \oplus K_{\Psi_3}$*.*

Complexity issues

The claims in part b) refer to solutions of the following systems:

$$(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2) \circ \Phi^i = 1,$$

(to be solved for 3 consecutive values of i, e.g., $i = -1, 0, 1$), and

$$(\alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_3 p_3^2 + \alpha_4 m_1^2 + \alpha_5 m_2^2 + \alpha_6 m_3^2) \circ \Phi^i = m_1 p_1 \circ \Phi^i,$$

etc. (to be solved for 6 consecutive values of i, e.g., $i \in [-2, 3]$).

This is a serious challenge for symbolic computations (for Φ^3 we are dealing with polynomials of degree 216 in 6 variables which is prohibitively complex). Various tricks invented to reduce the range of i.

Integral for non-integrable Kahan discretizations

- E. Celledoni, R.I. McLachlan, B. Owren, G.R.W. Quispel. *Geometric properties of Kahan's method.* J. Phys. A, 2013, **46**, 025201.

Theorem. *Let* $f(x) = J\nabla H(x)$*, with* $J \in so(n)$*, Hamilton function* $H : \mathbb{R}^n \to \mathbb{R}$ *of* $\deg = 3$*. Then* $\Phi_f(x, \epsilon)$ *admits a rational integral:*

$$\widetilde{H}(x, \epsilon) = H(x) + \frac{\epsilon}{3}(\nabla H(x))^{\mathrm{T}} \left(I - \frac{\epsilon}{2} f'(x) \right)^{-1} f(x),$$

and an invariant volume form

$$\frac{dx_1 \wedge \ldots \wedge dx_n}{\det \left(I - \frac{\epsilon}{2} f'(x) \right)}.$$

Degree of denominator $\det(I - \frac{\epsilon}{2} f'(x))$ *is n, degree of numerator of* $\widetilde{H}(x, \epsilon)$ *is* $n + 1$*.*

. Part 2. Integrability of planar quadratic birational maps

Why planar?

- Planar algebraic geometry is much simpler.
- Structure of the group of birational maps of $\mathbb{P}^n$ is unknown for $n \geq 3$. For $n = 2$, generated by quadratic maps (M. Noether theorem).
- For $n \geq 3$, many new phenomena. For instance, there does not hold necessarily that $\deg \Phi^{-1} = \deg \Phi$. (Kahan maps have this property and thus are very special!)

Planar birational maps

- Consider a birational map

$$\phi\colon \mathbb{CP}^2 \to \mathbb{CP}^2, \quad [x:y:z] \mapsto [X:Y:Z],$$

X, Y, Z homogeneous polynomials of deg $= d$ without a non-trivial (polynomial) common factor.

- *Indeterminacy set* (finitely many points, are blown up by ϕ):

$$\mathcal{I}(\phi) = \{X = Y = Z = 0\}.$$

- *Critical set* (dim $= 1$, is blown down by ϕ):

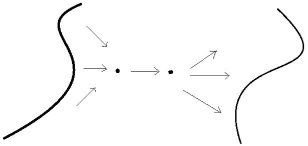
$$\mathcal{C}(\phi) = \{\det \partial(X, Y, Z)/\partial(x, y, z) = 0\}.$$

Degree lowering and singularity confinement

A component $V \subset \mathcal{C}(\phi)$ is a *degree lowering curve*, if for some $n \in \mathbb{N}$ we have $\phi^n(V) \in \mathcal{I}(\phi)$. A *singularity confinement pattern* is a sequence

$$\mathcal{C}(\phi) \supset V \to \phi(V) \to \cdots \to \phi^n(V) \to \phi^{n+1}(V) \subset \mathcal{C}(\phi^{-1}).$$

A presence of such a curve is necessary and sufficient for $\deg(\phi^n) < (\deg \phi)^n$.



Algebraic entropy

Definition. *Dynamical degree* and *algebraic entropy* of ϕ are

$$\lambda_1(\phi) = \lim_{n\to\infty} (\deg(\phi^n))^{1/n} \le d \quad \text{and} \quad h(\phi) = \log(\lambda_1(\phi)) \le \log(d).$$

Inequalities strict iff there exist degree lowering curves.

How drastic can be the degree drop of iterations ϕ^n?

Definition. A birational map ϕ is *integrable* if $h(\phi) = 0$.

Birational quadratic maps of $\mathbb{P}^2$

A generic birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of $\deg = 2$ can be represented as $\phi = A_1 \circ \sigma \circ A_2$, where $A_1, A_2 \in \mathrm{Aut}(\mathbb{P}^2)$, and

$$\sigma : [x : y : z] \to [yz : xz : xy].$$

The dimension of this orbit is 14.

A generic map from this set, not an involution, can be described by a pair of bilinear (Kahan type) relations:

$$\widetilde{x} - x = a_1 + a_2(x + \widetilde{x}) + a_3(y + \widetilde{y}) + a_4 x\widetilde{x} + a_5 y\widetilde{y} + a_6 x\widetilde{y} + a_7 y\widetilde{x},$$
$$\widetilde{y} - y = b_1 + b_2(x + \widetilde{x}) + b_3(y + \widetilde{y}) + b_4 x\widetilde{x} + b_5 y\widetilde{y} + b_6 x\widetilde{y} + b_7 y\widetilde{x}.$$

Singularities of birational quadratic maps of $\mathbb{P}^2$

- Singularities: $\mathcal{I}(\phi) = \{p_1, p_2, p_3\}$, $\mathcal{I}(\phi^{-1}) = \{q_1, q_2, q_3\}$.
- ϕ blows up points p_1, p_2, p_3 to lines $(q_2 q_3), (q_1 q_3), (q_1 q_2)$, resp.
- ϕ blows down lines $(p_2 p_3), (p_1 p_3), (p_1 p_2)$ to points q_1, q_2, q_3, resp.

Lifting to automorphism

Definition. Map ϕ is *confining*, if all three lines $(p_j p_k)$ are *degree lowering* (i.e., participate in *singularity confinement patterns*):

$$(p_j p_k) \to q_i \to \phi(q_i) \to \cdots \to \phi^{n_i - 1}(q_i) = p_{\sigma_i} \to (q_{\sigma_j} q_{\sigma_k}).$$

Orbit data of a confining ϕ consist of (n_1, n_2, n_3), $(\sigma_1, \sigma_2, \sigma_3)$.

A confining map ϕ can be lifted to an automorphism $\hat{\phi}$ of a surface S obtained from $\mathbb{P}^2$ by blowing up all participating points.

Dynamical degree $\lambda_1(\phi)$ can be found as the spectral radius of the action of $\hat{\phi}^*$ on $\mathrm{Pic}(S)$.

Theorem [Bedford, Kim' 2004]. For a confining map, $\lambda_1(\phi)$ depends only on the orbit data associated to ϕ.

Example of integrable planar birational map: Kahan discretization of Hamiltonian systems

For $n = 2$, consider $f(x, y) = J\nabla H(x, y)$, with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Φ_f is a birational planar map with an invariant measure and an integral $\Rightarrow$ completely integrable. Integral:

$$\widetilde{H}(x, y, \epsilon) = \frac{C(x, y, \epsilon)}{D(x, y, \epsilon)},$$

where $\deg C = 3$, $\deg D = 2$. Level sets:

$$\mathcal{E}_\lambda = \big\{(x, y) : C(x, y, \epsilon) - \lambda D(x, y, \epsilon) = 0\big\},$$

a pencil of cubic curves, characterized by its nine *base points*. On each invariant curve, Φ_f induces a translation (respective to the addition law on this curve).

Complexification, projectivization

Pencil

$$\bar{\mathcal{E}}_\lambda = \Big\{[x:y:z] \in \mathbb{CP}^2 \,:\, \bar{C}(x,y,z,\epsilon) - \lambda z \bar{D}(x,y,z,\epsilon) = 0\Big\}.$$

spanned by two curves,

$$\bar{\mathcal{E}}_0 = \Big\{[x:y:z] \in \mathbb{CP}^2 \,:\, \bar{C}(x,y,z,\epsilon) = 0\Big\},$$

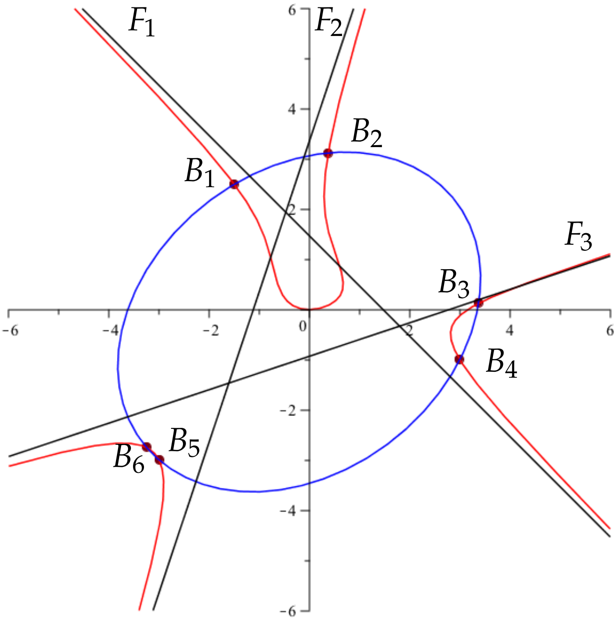
assumed nonsingular, and

$$\bar{\mathcal{E}}_\infty = \Big\{[x:y:z] \in \mathbb{CP}^2 \,:\, z\bar{D}(x,y,z,\epsilon) = 0\Big\}$$

reducible, consisting of conic $\{\bar{D}(x,y,z,\epsilon) = 0\}$ and the line at infinity $\{z = 0\}$. Three base points at infinity:

$$\{F_1, F_2, F_3\} = \bar{\mathcal{E}}_0 \cap \{z = 0\},$$

and six further base points $\{B_1, \ldots B_6\} = \bar{\mathcal{E}}_0 \cap \{\bar{D} = 0\}$.

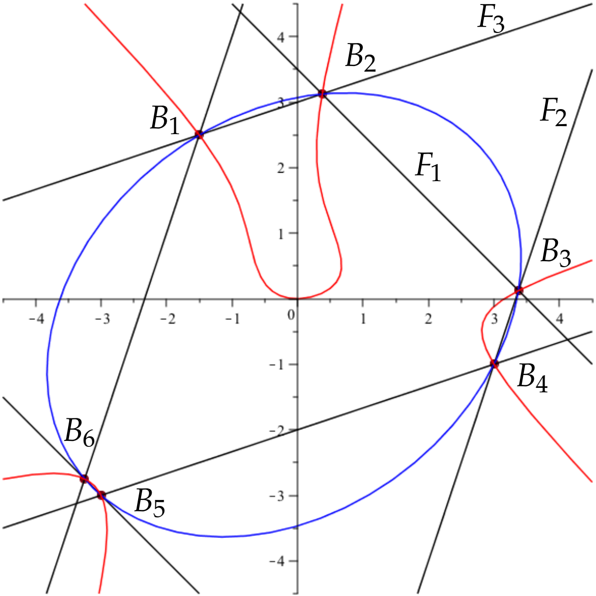


F_1
F_2
F_3
B_1
B_2
B_3
B_4
B_5
B_6
6
4
2
0
-2
-4
-6
-6
-4
-2
2
4
6

Main result

- M. Petrera, J. Smirin, Yu. S. *Geometry of the Kahan discretizations of planar quadratic Hamiltonian systems*. Proc. R. Soc. A **476** (2019) 20180761

Theorem. *A pencil of elliptic curves consists of invariant curves for Kahan's discretization of a planar quadratic Hamiltonian vector field iff the hexagon through the six finite base points has three pairs of parallel sides which pass through the three base points at infinity.*



F_3
B_2
F_2
B_1
F_1
B_3
B_4
B_6
B_5
4
3
2
1
0
-1
-2
-3
-4

Manin involutions for cubic curves

Definition. Consider a nonsingular cubic curve $\bar{\mathcal{E}}$ in $\mathbb{CP}^2$.

- For a point $P_0 \in \bar{\mathcal{E}}$, the *Manin involution* $I_{\bar{\mathcal{E}},P_0} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is defined as follows:
  - For $P \neq P_0$, the point $\bar{P} = I_{\bar{\mathcal{E}},P_0}(P)$ is the unique third intersection point of $\bar{\mathcal{E}}$ with the line $(P_0 P)$;
  - For $P = P_0$, the point $\bar{P} = I_{\bar{\mathcal{E}},P_0}(P)$ is the unique second intersection point of $\bar{\mathcal{E}}$ with the tangent line to $\bar{\mathcal{E}}$ at $P = P_0$.
- For two distinct points $P_0, P_1 \in \bar{\mathcal{E}}$, the *Manin transformation* $M_{\bar{\mathcal{E}},P_0,P_1} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is defined as

$$M_{\bar{\mathcal{E}},P_0,P_1} = I_{\bar{\mathcal{E}},P_1} \circ I_{\bar{\mathcal{E}},P_0}.$$

With a natural addition law on $\bar{\mathcal{E}}$:

$$I_{\bar{\mathcal{E}},P_0}(P) = -(P_0 + P), \quad M_{\bar{\mathcal{E}},P_0,P_1}(P) = P + P_0 - P_1.$$

Manin involutions for cubic pencils

Definition. Consider a pencil $\mathfrak{E} = \{\bar{\mathcal{E}}_\lambda\}$ of cubic curves in $\mathbb{CP}^2$ with at least one nonsingular member.

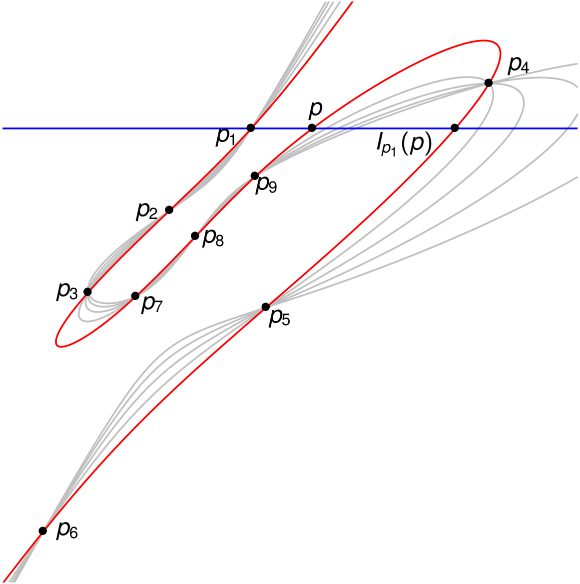
- Let B be a base point of the pencil. The *Manin involution* $I_{\mathfrak{E},B} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is a birational map defined as follows. For any $P \in \mathbb{CP}^2$, not a base point of $\mathfrak{E}$, let $\bar{\mathcal{E}}_\lambda$ be the unique curve of $\mathfrak{E}$ through P. Set

$$I_{\mathfrak{E},B}(P) = I_{\bar{\mathcal{E}}_\lambda,B}(P).$$

- Let B_1, B_2 be two distinct base points of the pencil. The *Manin transformation* $M_{\mathfrak{E},B_1,B_2} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is a birational map defined as

$$M_{\mathfrak{E},B_1,B_2} = I_{\mathfrak{E},B_2} \circ I_{\mathfrak{E},B_1}.$$

Manin involutions for cubic pencils



Direct statement. Proof.

First one shows tha Kahan map Φ_f is a Manin transformation in six different ways:

$$\begin{aligned}\Phi_f &= I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},F_1} = I_{\mathfrak{E},F_1} \circ I_{\mathfrak{E},B_4} \\ &= I_{\mathfrak{E},B_5} \circ I_{\mathfrak{E},F_2} = I_{\mathfrak{E},F_2} \circ I_{\mathfrak{E},B_2} \\ &= I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},F_3} = I_{\mathfrak{E},F_3} \circ I_{\mathfrak{E},B_6}.\end{aligned}$$

Thus (on any invariant curve of $\mathfrak{E}$):

$$F_1 - B_1 = B_2 - F_2 = F_3 - B_3 = B_4 - F_1 = F_2 - B_5 = B_6 - F_3,$$

and

$$F_1 + F_2 + F_3 = O.$$

Have, e.g.:

$$B_1 + B_2 = F_1 + F_2 = -F_3 \quad \Leftrightarrow \quad B_1 + B_2 + F_3 = O.$$

Thus, line $(B_1 B_2)$ passes through F_3. □

Inverse statement. Proof.

Prescribe arbitrary nine coefficients of the side lines of the hexagon (three slopes μ_1, μ_2, μ_3 and six heights $b_1, \ldots, b_6$):

$$\begin{aligned} (B_1B_2): y &= \mu_3 x + b_1, & (B_4B_5): \ y &= \mu_3 x + b_4, \\ (B_2B_3): y &= \mu_1 x + b_2, & (B_5B_6): \ y &= \mu_1 x + b_5, \\ (B_3B_4): y &= \mu_2 x + b_3, & (B_6B_1): \ y &= \mu_2 x + b_6. \end{aligned}$$

This defines nine points $B_1, \ldots, B_6$ and F_1, F_2, F_3, therefore a pencil $\mathfrak{E}$ of cubic curves with those nine base points. Set

$$\begin{aligned} \Phi &= I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},F_1} = I_{\mathfrak{E},F_1} \circ I_{\mathfrak{E},B_4} \\ &= I_{\mathfrak{E},B_5} \circ I_{\mathfrak{E},F_2} = I_{\mathfrak{E},F_2} \circ I_{\mathfrak{E},B_2} \\ &= I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},F_3} = I_{\mathfrak{E},F_3} \circ I_{\mathfrak{E},B_6}. \end{aligned}$$

This is a birational map of $\mathbb{CP}^2$ of degree 2. Check that this is a Kahan discretization of $f = J\nabla H$ with deg $H = 3$.

Inverse statement. Proof.

Explicit expression:

$$H(x,y) =$$

$$\frac{2\mu_{12}}{b_{14}\mu_{23}\mu_{13}}\Big(\tfrac{1}{3}(\mu_3 x - y)^3 + \tfrac{1}{2}(b_1+b_4)(\mu_3 x - y)^2 + b_1 b_4(\mu_3 x - y)\Big)$$

$$-\frac{2\mu_{23}}{b_{25}\mu_{12}\mu_{13}}\Big(\tfrac{1}{3}(\mu_1 x - y)^3 + \tfrac{1}{2}(b_2+b_5)(\mu_1 x - y)^2 + b_2 b_5(\mu_1 x - y)\Big)$$

$$+\frac{2\mu_{13}}{b_{36}\mu_{12}\mu_{23}}\Big(\tfrac{1}{3}(\mu_2 x - y)^3 + \tfrac{1}{2}(b_3+b_6)(\mu_2 x - y)^2 + b_3 b_6(\mu_2 x - y)\Big),$$

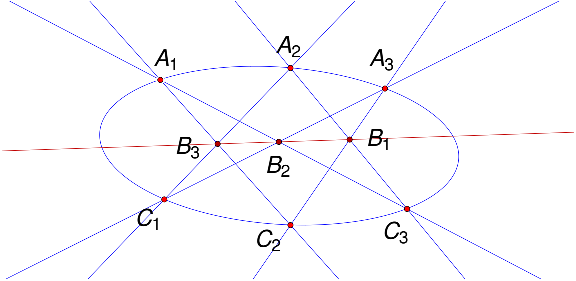
where $b_{ij} = b_i - b_j$, $\mu_{ij} = \mu_i - \mu_j$.

Geometry implies dynamics! □

Projective generalization of Hamiltonian case

Pascal configuration: six points A_1, A_2, A_3, C_1, C_2, C_3 on a conic $\mathcal{C}$, and three intersection points on a line ℓ:

$$B_1 = (A_2C_3)\cap(A_3C_2), \quad B_2 = (A_3C_1)\cap(A_1C_3), \quad B_3 = (A_1C_2)\cap(A_2C_1).$$



Consider pencil $\mathfrak{E}$ of cubic curves passing through the nine points A_i, C_i, B_i (contains a reducible cubic $\mathcal{C} \cup \ell$).

Construction

Theorem [S.' 2020]. *The map*

$$\begin{aligned}\Phi &= I_{\mathfrak{C},A_1}\circ I_{\mathfrak{C},B_1} = I_{\mathfrak{C},B_1}\circ I_{\mathfrak{C},C_1}\\ &= I_{\mathfrak{C},A_2}\circ I_{\mathfrak{C},B_2} = I_{\mathfrak{C},B_2}\circ I_{\mathfrak{C},C_2}\\ &= I_{\mathfrak{C},A_3}\circ I_{\mathfrak{C},B_3} = I_{\mathfrak{C},B_3}\circ I_{\mathfrak{C},C_3}\end{aligned}$$

is a birational map **of degree** 2 *with*

- $\mathcal{I}(\Phi)=\{C_1,C_2,C_3\}$, *blown up to lines* $c_1=(A_2A_3)$, $c_2=(A_3A_1)$, $c_3=(A_1A_2)$,
- $\mathcal{C}(\Phi)$ *consisting of three lines* $a_1=(C_2C_3)$, $a_2=(C_3C_1)$, $a_3=(C_2C_3)$, *blown down to points* A_1, A_2, A_3.

Singularity confinement patterns of the map Φ*:*

$$\begin{aligned}(C_2C_3)\to A_1\to B_1\to C_1\to (A_2A_3),\\ (C_3C_1)\to A_2\to B_2\to C_2\to (A_3A_1),\\ (C_1C_2)\to A_3\to B_3\to C_3\to (A_1A_2).\end{aligned}$$

Proof

To show: why the six Manin transformations correspond to one and the same translation on any curve of the pencil:

$$A_1 - B_1 = B_1 - C_1 = A_2 - B_2 = B_2 - C_2 = A_3 - B_3 = B_3 - C_3.$$

Collinearities of Pascal configuration are translated to:

$$\begin{aligned} A_2 + C_3 + B_1 &= O, \quad & A_3 + C_2 + B_1 &= O, \\ A_3 + C_1 + B_2 &= O, \quad & A_1 + C_3 + B_2 &= O, \\ A_1 + C_2 + B_3 &= O, \quad & A_2 + C_1 + B_3 &= O, \end{aligned}$$

and

$$B_1 + B_2 + B_3 = O.$$

$$\text{Now:}\quad \begin{aligned} A_1 + C_1 &= -(C_2 + B_3) - (A_3 + B_2) \\ &= -(A_3 + C_2) - (B_2 + B_3) \;=\; B_1 + B_1, \end{aligned}$$

which proves that $A_1 - B_1 = B_1 - C_1$. Similarly,

$$A_2 + C_1 \;=\; -B_3 \;=\; B_1 + B_2,$$

which proves that $B_1 - C_1 = A_2 - B_2$.

All other equations follow in the same way. □

An early example

R. Penrose, C. Smith. *A quadratic mapping with invariant cubic curve*. Math. Proc. Camb. Phyl. Soc. **89** (1981), 89–105:

$$\Phi : \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_0(x_0 + ax_1 + a^{-1}x_2) \\ x_1(x_1 + ax_2 + a^{-1}x_0) \\ x_2(x_2 + ax_0 + a^{-1}x_1) \end{bmatrix}$$

with

$$A_1 = [0 : 1 : -a], \quad C_1 = [0 : a : -1], \quad B_1 = [0 : 1 : -1]$$

(and others cyclically). Upon a projective transformation sending B_1, B_2, B_3 to infinity, get a Kahan discretization of a Hamiltonian vector field with $H(x, y) = xy(1 - x - y)$ with the time step $\epsilon = (a - 1)/(a + 1)$.

Further examples: $(\gamma_1, \gamma_2, \gamma_3)$-family of 2d quadratic systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{\ell_1^{\gamma_1 - 1} \ell_2^{\gamma_2 - 1} \ell_3^{\gamma_3 - 1}} J \nabla H,$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H(x, y) = (\ell_1(x, y))^{\gamma_1} (\ell_2(x, y))^{\gamma_2} (\ell_3(x, y))^{\gamma_3},$$

$\ell_i(x, y) = a_i x + b_i y$ are linear forms, and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$.

Origin: reduced Nahm equations for symmetric monopoles [Hitchin, Manton, Murray' 1995]

- Tetrahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$:

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = -2xy, \end{cases} \qquad H_1(x, y) = \frac{y}{3}(3x^2 - y^2).$$

- Octahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$:

$$\begin{cases} \dot{x} = x^2 - 6y^2, \\ \dot{y} = -3xy - 2y^2, \end{cases} \qquad H_2(x, y) = \frac{y}{2}(2x + 3y)(x - y)^2.$$

- Octahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$:

$$\begin{cases} \dot{x} = 2x^2 - y^2, \\ \dot{y} = -10xy + y^2, \end{cases} \qquad H_3(x, y) = \frac{y}{6}(3x - y)^2(4x + y)^3.$$

In all three cases all level sets $H_i(x, y) = c$ are elliptic curves.

The $(\gamma_1, \gamma_2, \gamma_3)$-family: discretization

Hirota-Kimura-Kahan discretizations are integrable [Petrera, Pfadler, S.' 2011]:

$$\begin{cases} \widetilde{x} - x = \epsilon(\widetilde{x}x - \widetilde{y}y), \\ \widetilde{y} - y = -\epsilon(\widetilde{x}y + x\widetilde{y}), \end{cases}$$

$$\begin{cases} \widetilde{x} - x = \epsilon(2\widetilde{x}x - 12\widetilde{y}y), \\ \widetilde{y} - y = -\epsilon(3\widetilde{x}y + 3x\widetilde{y} + 4\widetilde{y}y), \end{cases}$$

$$\begin{cases} \widetilde{x} - x = \epsilon(2\widetilde{x}x - \widetilde{y}y), \\ \widetilde{y} - y = \epsilon(-5\widetilde{x}y - 5x\widetilde{y} + \widetilde{y}y). \end{cases}$$

In all three cases, the map admits an invariant pencil of elliptic curves, of degrees 3, 4, and 6, respectively.

The $(\gamma_1, \gamma_2, \gamma_3)$-family: classification of integrable cases through discretization

Theorem [Zander' 2020]. *The only three cases when the Kahan discretization of the* $(\gamma_1, \gamma_2, \gamma_3)$*-system is confining, are* $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1), (1, 1, 2)$*, and* $(1, 2, 3)$*. The orbit data in these three cases are:* $(\sigma_1, \sigma_2, \sigma_3) = (1, 2, 3)$ *and, respectively,*

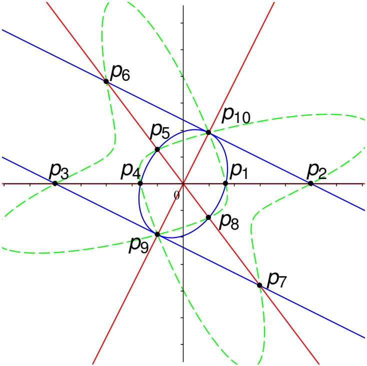
$$(n_1, n_2, n_3) = (3, 3, 3),\ (4, 4, 2),\ \textit{and}\ \ (6, 3, 2).$$

Observe: these (n_1, n_2, n_3) are the only positive integer solutions of

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1.$$

Puzzle: what do lengths of singularity confinement patterns have to do with tilings of the plane by congruent triangles???

Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$



Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$

- Invariant pencil consists of quartic curves with two double points: $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$.
- $\mathcal{I}(\phi) = \{p_4, p_8, p_{10}\}$, $\mathcal{I}(\phi^{-1}) = \{p_1, p_5, p_9\}$.
- Singularity confinement patterns:

$$
\begin{aligned}
&(p_8 p_{10}) \to p_1 \to p_2 \to p_3 \to p_4 \to (p_5 p_9) \\
&(p_4 p_{10}) \to p_5 \to p_6 \to p_7 \to p_8 \to (p_1 p_9) \\
&(p_4 p_8) \to p_9 \to p_{10} \to (p_1 p_5)
\end{aligned}
$$

- What is the geometric representation?

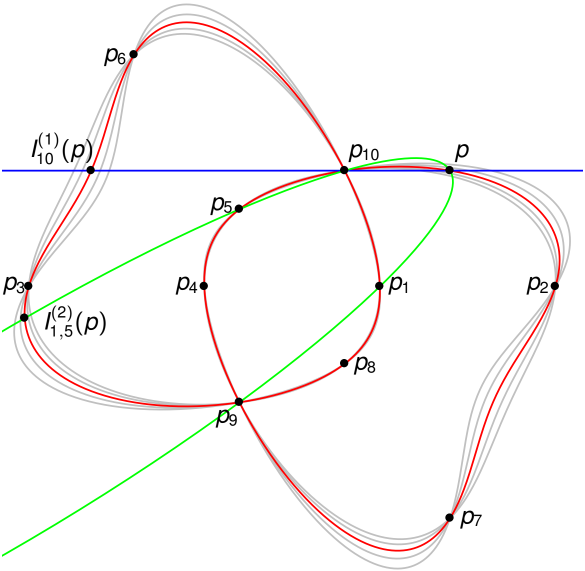
Involutions for quartic pencils with two double points

Manin involutions for $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$:

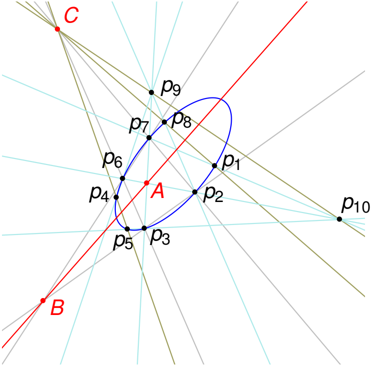
- $I_k^{(1)}$, $k \in \{9, 10\}$: $I_k^{(1)}(p)$ is the third intersection point of the quartic through p with the line (pp_k).
- $I_{i,j}^{(2)}$, $i, j \in \{1, \ldots, 8\}$: $I_{i,j}^{(2)}(p)$ is the sixth intersection point of the quartic through p with the conic through p_9, p_{10}, p_i, p_j, p.

Are derived from Manin involutions for a cubic pencil upon a quadratic Cremona transformation resolving both double points.

Involutions for quartic pencils with two double points



Quadratic Manin maps for special quartic pencils



Geometry of base points of a *projectively symmetric quartic pencil with two double points* $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$.

Quadratic Manin maps for special quartic pencils

Theorem [Petrera, S., Wei, Zander' 2021].

1. *The projective involution* σ *can be represented as*

$$\sigma = I_{1,8}^{(2)} = I_{2,7}^{(2)} = I_{3,6}^{(2)} = I_{4,5}^{(2)}.$$

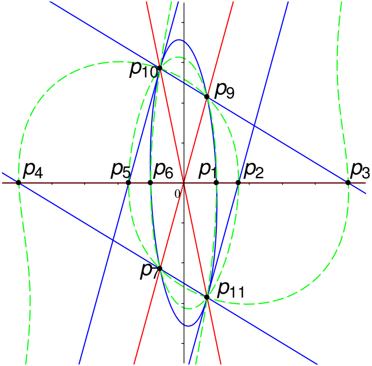
2. *The map*

$$\phi \;=\; I_{i,k}^{(2)} \circ I_{j,k}^{(2)} \;=\; I_9^{(1)} \circ \sigma \;=\; \sigma \circ I_{10}^{(1)},$$

$(i,j) \in \{(1,2),(2,3),(3,4),(5,6),(6,7),(7,8)\}$ *and* $k \in \{1,\ldots,8\}$ *distinct from* i,j*, is a birational map* **of degree 2**, *with the singularity confinement patterns:*

$$(p_8 p_{10}) \to p_1 \to p_2 \to p_3 \to p_4 \to (p_5 p_9),$$
$$(p_4 p_{10}) \to p_5 \to p_6 \to p_7 \to p_8 \to (p_1 p_9),$$
$$(p_4 p_8) \to p_9 \to p_{10} \to (p_1 p_5).$$

Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$



Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$

- Invariant pencil of sextic curves with 3 double points and 2 triple points: $\mathfrak{C} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$.
- $\mathcal{I}(\phi) = \{p_6, p_9, p_{11}\}, \mathcal{I}(\phi^{-1}) = \{p_1, p_7, p_{10}\}$.
- Singularity confinement patterns:

$$(p_9 p_{11}) \to p_1 \to p_2 \to p_3 \to p_4 \to p_5 \to p_6 \to (p_7 p_{10}),$$
$$(p_6 p_{11}) \to p_7 \to p_8 \to p_9 \to (p_1 p_{10}),$$
$$(p_6 p_9) \to p_{10} \to p_{11} \to (p_1 p_7).$$

- What is the geometric representation?

Kahan discretization for $(\gamma_1,\gamma_2,\gamma_3)=(1,2,3)$

Manin involutions for $\mathfrak{E}=\mathcal{P}(6;p_1,\ldots,p_6,p_7^2,p_8^2,p_9^2,p_{10}^3,p_{11}^3)$:

- $I_{i,j,k}^{(4)}$, $i,j\in\{1,\ldots,6\}$, $k\in\{7,8,9\}$: e.g., $I_{i,j,9}^{(4)}$ is defined in terms of intersection of $\mathfrak{E}$ with quartics of the pencil

$$\mathcal{P}(4;p_i,p_j,p_7,p_8,p_9^2,p_{10}^2,p_{11}^2).$$

- $I_{i,k}^{(3)}$, $i\in\{1,\ldots,6\}$, $k\in\{10,11\}$: e.g., $I_{i,10}^{(3)}$ is defined in terms of intersection of $\mathfrak{E}$ with cubics of the pencil

$$\mathcal{P}(3;p_i,p_7,p_8,p_9,p_{10}^2,p_{11}).$$

Theorem [Petrera, S, Wei, Zander' 2021]. *The map ϕ can be represented as compositions of (suitably defined) Manin involutions in the following ways:*

$$\phi=I_{i,k,m}^{(4)}\circ I_{j,k,m}^{(4)}=I_{i,n}^{(3)}\circ I_{j,n}^{(3)}$$

for any $(i,j)\in\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$, $k\in\{1,\ldots,6\}\setminus\{i,j\}$, and $m\in\{7,8,9\}$, $n\in\{10,11\}$.

Conclusions, work in progress and open problems

- Classification of integrable cases of Kahan discretization for the $(\gamma_1, \gamma_2, \gamma_3)$-family.
- Geometric construction of Manin involutions for pencils of elliptic curves of degree 4 and 6.
- Integrable Kahan discretizations for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$ are Manin maps for pencils of elliptic curves of degree 3, 4, 6, resp.
- Special geometry of base points ensures deg $= 2$ for certain Manin maps.
- Work in progress: singularity structure and geometric description for higher-dimensional examples, e.g., Kahan discretization of the Euler top (3D, $g = 1$) or the Clebsch system (6D, $g = 2$).